STRESS DIFFUSION IN PARTIALLY DEBONDED PULL-OUT MODEL

A Thesis Submitted
In Partial Fulfilment of the Requirements
For the Degree of
MASTER OF TECHNOLOGY

By
SUDHIR KAMLE

DEPARTMENT OF AERONAUTICAL ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

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Sudhir Kamle

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LIST OF SYMBOLS

c	=	Non-dimensional crack length
f_1	=	Non-dimensional axial stress applied at fiber end
l	=	Non-dimensional fiber length
R	=	Non-dimensional radius vector
XDB	=	Non-dimensional axial location of debonded region
z	=	Non-dimensional axial vector
u_f, w_f	=	Non-dimensional radial and axial displacement of fiber
u_m, w_m	=	Non-dimensional radial and axial displacement of matrix
NF, MF, K	=	Number of terms in infinite series
X_n^i, X_{mn}^i	=	Coefficients of Fourier-series (See Appendix I)
G_n^i, G_{mn}^i	=	Coefficients of Fourier-Bessel series (See Appendix I)
SKNWN, SALPH, SBETA, SCONST	=	Column vectors associated with unknowns
EKNWN, EALPH, EBETA	=	Column vectors associated with unknowns
A_F^1, A_F^2, ... etc.	=	Matrices associated with equations on fiber
B_M^1, B_M^2, ... etc.	=	Matrices associated with equations on matrix
σ_r^f, σ_z^f, σ_{rz}^f	=	Non-dimensional radial, axial and shear stress in the fiber

$\sigma_r^m, \sigma_z^m, \sigma_{rz}^m$	=	Non-dimensional radial, axial and shear stress in the matrix
λ_f, λ_m	=	Non-dimensional Lame coefficients for fiber and matrix
	=	$\frac{2\nu}{(1 - 2\nu)}$
ν_f, ν_m	=	Poisson's ratio for fiber and matrix
μ_f	=	Non-dimensional shear modulus of fiber
	=	Ratio of shear modulus of fiber to matrix
γ_F	=	Ratio of Young's modulus of Fiber to matrix in symbol table

SYNOPSIS

The present analysis is aimed at studying the growth of a debonding crack in a composite. The model considered is a single fiber embedded in a cylindrical matrix with tensile load being applied to the fiber end. The fiber is assumed to have been bonded to the matrix everywhere except in a small region. Making use of the general solution for axisymmetric cylinders, a system of linear equations is developed. These are then solved for stress and displacement fields. The results have all been presented in a non-dimensional graphical form.

1. INTRODUCTION & LITERATURE SURVEY

1.1 Introduction

The word composite means 'made up of two or more distinct parts'. From the earliest uses, the goals for composite development have been to achieve a combination of properties not achievable by any of the elemental materials acting alone; thus a solid could be prepared from constituents which, by themselves could not satisfy a particular design requirement. The struggle to develop new and better composites has not been limited to mankind, even nature has ventured into it and produced bamboo with excellent impact strength and buckling strength and bone with strength and lightness, to name only a few.

Some examples of man-made composite materials are cermets, glass fiber reinforced plastics, reinforced concrete and plywood. Cermets (composites of ceramic and metal) are used in cutting tools, turbine parts and as nuclear fuel elements and control rods. Glass-fiber reinforced plastics are being used in air-conditioners and humidifers, fire extinguishers, blood testing equipment. Steel fibers with matrix as unsaturated polyesters are used in chemical plants for polyester piping, ductwork, fans etc. and in building industry as thermosetting decorative sheet. And to top it all, composite materials have found their way in the more

strenuous and demanding roles like transportation. CFR epoxies have been used in DOT Fairchild experimental safety car, while DC-10 and Lockheed L-1011 airplanes have used a hetrocyclic (aromatic) polymer named PR9-49. Looking at the tremendous studies made in the area of composite materials, the day does not seem far off when metals will be completely replaced by composites, possibly even in coinage.

1.2 Literature Survey

The majority of composites used in structural applications fall in the fiber-reinforced category. The basic concept behind fiber reinforcement is the production of a two-phase composite structure in which deformation of the matrix is used to transfer stress, by means of shear tractions at the fiber matrix interface, to the embedded, high strength fibers. Provided the length of the fibers is sufficient, they should then be constrained to take up the same deformation as the matrix over the greater part of their length and thus effectively reinforce the matrix. In addition, the presence of the fibers retard the propagation of cracks and thus produce a material which is tough as well as of high strength. The present literature survey is concerned principally with the basic concepts of fiber reinforcement and the discussion of various theories proposed to predict the thermoelastic properties of fibrous composites. All

these theories can be classified as belonging to either one of the following: netting analysis, mechanics of materials, self consistent model, variational, exact (within the context of classical elasticity), statistical, discrete element, semi-empirical methods and theories accounting for microstructure. Certain assumptions common to most of these theories are: the matrix and fiber are linearly elastic, homogeneous and free of voids; there is complete bonding at the interface of the constituents and there is no transitional region between them, the ply is initially in a stress free state and the fibers are regularly spaced and aligned.

Netting analysis assumes that the filaments provide all the longitudinal stiffness and the matrix provides the shear and transverse stiffness and the Poisson effect. This method provides acceptable values for longitudinal properties and low values for transverse and shear strength. In mechanics of materials approach, the ply's averaged stress-strain temperature state is expressed in terms of the stress-strain temperature states of its constituents using displacement, continuity and force equilibrium conditions. Substitution of the ply averaged stress-strain-thermal state in the appropriate definitions for thermoelastic properties yields the desired results.

Hill[4] modeled the composite as a single fiber embedded in an unbounded homogeneous medium which is

macroscopically indistinguishable from the composite. The medium is subjected to a uniform loading at infinity which induces a uniform strain field in the filament. This strain field is used to estimate elastic constants. Hill's method, coming under the self-consistent model approach gives reliable values at low filament volume ratios, reasonable values at intermediate ratios and unreliable values at high ones.

Rosen [4] has used variational method to obtain bounds in the ply's elastic properties. The minimum complementary energy theorem and the minimum potential energy theorem have been used to obtain lower and upper bounds respectively.

An exact method of analyzing filamentary composites consists of assuming that the fibers are arranged in a regular periodic array. The resulting elasticity problem is solved either by a series development or by some numerical scheme. Sarma [9] has solved for stresses in a fiber-pullout model using exact method. Statistical methods assume that the composite is a homogeneous solid whose thermoelastic properties vary randomly with position. One fluctuating term is added to the field equations. The resulting equations are statistically averaged to give the desired results.

Discrete element methods use a finite element scheme (usually a constant-stress triangular element is employed) to solve for elastic constants. Semi-empirical methods tend to retain the physical complexities of the

problem while relaxing the mathematical rigour. Bolotin has proposed a theory accounting for microstructure assuming the fiber to be more deformation resistant than the matrix; the distance between fiber and matrix and thickness of fiber being small compared with the characteristic dimensions of the body and comparable with the distances over which the functions characterizing the stress-strain state of the reinforcing elements change appreciably. This equations resemble the ones for a Voigt-Cosserat medium, that is, a medium for which the stress tensor is not symmetric.

Research work on composite materials has primarily been centered around models assuming no solubility or no reaction at the interface. Also properties were not associated with the interface, per se[6] For example, the term 'interfacial shear strength' was often applied to the stress in the matrix immediately adjacent to the filament. A further assumption was made that the interface was stronger than the matrix so that the matrix flow limited load transfer from and to the filaments. More practical systems, now in vogue have used higher strength matrices, causing the failure path to move to the interface and this has led to the development of theories for weak interfaces.

Cooper and Kelly [2] have divided the mechanical properties of composites into those affected by the tensile strength of the interface σ_i, and those dependent on the shear

strength τ_i. For longitudinal tensile loading, they conclude that the tensile strength of the interface is not critical, but the interface shear strength controls the following properties, critical or load transfer length i.e. the filament length required for the longitudinal stress in the filament to reach its fracture stress; composite fracture under conditions of fiber pullout and the deformation of the matrix in fracture. Ebert and Gadd [3] have modeled the composite by considering a typical fiber-matrix unit to have a cylindrical configuration with a core of matrix and an outer sleeve of fiber. They conclude that the transverse stresses generated in the composite reach their maximum intensity at the fiber matrix interface. Lawrence [7] has discussed the distribution of load and shear stress along the fiber length and the conditions for fiber pullout. He concludes that the maximum fiber load depends on length of embedded fiber and ratio between the shear strength and the frictional shear strength of the interface and that the development of debonding of the fiber from the matrix can have a marked effect on the maximum shear stress developed at the interface.

1.3 Scope of the Present Work

The present work has grown out of the realization that the majority of work done in the past has started either from the assumption of a perfect or no bonding at

the interface. The most important cases of limited strength bonds, or regions of good bonding mixed with regions which are unbonded have received little attention. It is the problem of regions of good bonding mixed with unbonded regions that interests us here. This problem is an extension of results obtained from Sarma[9] The elastic solutions as obtained by Murthy [8] has been used for numerically evaluating the stress and displacement fields.

2. ELASTIC ANALYSIS

2.1 Choice of the Model

In the present problem, the model considered is a single filament embedded in a cylindrical matrix. The choice of the model was primarily dictated by its resemblance (in geometry) to actual composites. Both the fiber and the matrix are assumed to be linearly elastic. The matrix is clamped on the lateral surface. Clamping is also done on the top surface of the model (see Figure 2.1). The load is applied to the bottom end of the fiber. The debonding is assumed to have occurred over a distance c at a place XDB units away from the top end.

Lame has solved the problem of determining the stress and strain fields in a circular elastic cylinder under various boundary and loading conditions. His solutions, however, assume a plane strain condition. For cylinders whose length is comparable to diameter, the plane strain condition is no longer valid.

2.2 Analysis of Elastic Cylinders

The stress displacement equations for an elastic, homogeneous medium in the presence of axisymmetric loading can be written in the form:

$$\sigma_{r\theta} = \sigma_{\theta z} = 0 \tag{1}$$

$$\sigma_r = (\lambda + 2\mu)\frac{\partial u}{\partial r} + \lambda\left(\frac{\partial w}{\partial z} + \frac{u}{r}\right) \tag{2}$$

$$\sigma_\theta = (\lambda + 2\mu)\frac{u}{r} + \lambda\left(\frac{\partial w}{\partial z} + \frac{\partial u}{\partial r}\right) \tag{3}$$

$$\sigma_z = (\lambda + 2\mu)\frac{\partial w}{\partial z} + \lambda\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) \tag{4}$$

$$\sigma_{rz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \tag{5}$$

where λ and μ are the Lame's constants.

The equilibrium equations to be satisfied by an axisymmetric problem are:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + R^* = 0 \tag{6}$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + Z^* = 0 \tag{7}$$

where R^* and Z^* represent the body forces per unit volume, in the r and z directions respectively.

Substitution for σ_r, σ_{rz} and σ_θ from Eqs. (1) to (5) in Eqs. (6) and (7) yields:

$$(\lambda + 2\mu)\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right) + \mu\frac{\partial^2 u}{\partial z^2} + (\lambda + \mu)\frac{\partial^2 w}{\partial r\,\partial z} + R^* = 0 \tag{8}$$

$$(\lambda + \mu)\left(\frac{\partial^2 u}{\partial r\,\partial z} + \frac{1}{r}\frac{\partial u}{\partial z}\right) + (\lambda + 2\mu)\frac{\partial^2 w}{\partial z^2} + \frac{\mu}{r}\frac{\partial w}{\partial r} + \mu\frac{\partial^2 w}{\partial r^2} + Z^* = 0 \tag{9}$$

Elimination of w from Eqs. (8) and (9) leads to the fourth order governing partial differential equation in r and z as:

$$\left(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r + \frac{\partial^2}{\partial z^2}\right)^2 u + \frac{1}{\mu}\frac{\partial^2 R^*}{\partial z^2} + \frac{1}{(\lambda + 2\mu)}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} r R^*$$

$$- \frac{(\lambda + \mu)}{\mu(\lambda + 2\mu)}\frac{\partial^2 Z^*}{\partial z \partial r} = 0 \qquad (10)$$

For the cases involving constant body forces, Eq. (10) reduces to:

$$\nabla_1^2 \nabla_1^2 u = \left(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} r + \frac{\partial^2}{\partial z^2}\right)^2 u = 0 \qquad (11)$$

Thus once the solution of Eq. (11) is known, w can be calculated using Eq. (8). The calculation of stresses and strains is then a simple exercise. This has already been done by Murthy. His results form the basis for the following analysis.

2.3 Displacement Equations for Fiber

The displacement equations for u^f and w^f (radial and axial displacements for fiber) are taken as:

$$u^f = (S_1 r^3 + S_3 r + S_5 r z^2 + T_1 r^3 z + T_5 r z^3)$$

$$+ \sum_{m=1}^{MF} r\, J_o(\delta_m r)\,[S_{1m}^{\delta} \cosh \delta_m z + T_{1m}^{\delta} \sinh \delta_m z]$$

$$+ \sum_{k=1}^{NSUB} \left[\frac{\cosh \beta_k z}{\cosh \beta_k l} S_{1k}^{\beta} + \frac{\sinh \beta_k z}{\sinh \beta_k l} T_{1k}^{\beta}\right] J_1(\beta_k r)$$

$$+ \sum_{n=1}^{NF} \left[\frac{I_1(\alpha_n r)}{I_1(\alpha_n)} S_{1n}^{\alpha} + \frac{rI_o(\alpha_n r)}{I_o(\alpha_n)} S_{3n}^{\alpha}\right] \cos \alpha_n z$$

$$+ \sum_{n=1}^{NF} \left[\frac{I_1(\alpha_n r)}{I_1(\alpha_n)} T_{1n}^{\alpha} + \frac{rI_o(\alpha_n r)}{I_o(\alpha_n)} T_{3n}^{\alpha}\right] \sin \alpha_n z \qquad (12)$$

$$w^f = - \frac{4(\lambda_f + 2\mu_f)}{(\lambda_f + \mu_f)} zr^2 S_1 + \frac{\mu_f}{(\lambda_f + \mu_f)} zr^2 S_5$$

$$- \frac{2(\lambda_f + 2\mu_f)}{(\lambda_f + \mu_f)} r^2 z^2 T_1 - \frac{3\mu_f}{2(\lambda_f + \mu_f)} T_5 r^2 z^2$$

$$+ \frac{1}{6}\left(\frac{4\mu_f}{\lambda_f + \mu_f} T_1 - \frac{3\lambda_f}{\lambda_f + \mu_f} T_5\right) z^4 + \frac{2z^3(4\mu_f S_1 - \lambda S_5)}{3(\lambda_f + \mu_f)}$$

$$+ \left[\frac{2\lambda_f + 3\mu_f}{4(\lambda_f + \mu_f)} T_1 + \frac{3}{16}\left(\frac{\lambda_f + 2\mu_f}{\lambda_f + \mu_f}\right) T_5\right] r^4$$

$$- \sum_{n=1}^{NF} \left[\frac{I_o(\alpha_n r)}{I_1(\alpha_n)} S_{1n}^{\alpha} + \left(\frac{rI_1(\alpha_n r)}{I_o(\alpha_n)}\right.\right.$$

$$\left.\left.+ \frac{2(\lambda_f + 2\mu_f)}{\alpha_n(\lambda_f + \mu_f)} \frac{I_o(\alpha_n r)}{I_o(\alpha_n)}\right) S_{3n}^{\alpha}\right] \sin \alpha_n z$$

$$+ \sum_{n=1}^{NF} \left[\frac{I_o(\alpha_n r)}{I_1(\alpha_n)} T_{1n}^{\alpha} + \left(\frac{rI_1(\alpha_n r)}{I_o(\alpha_n)}\right.\right.$$

$$\left.\left.+ \frac{2(\lambda_f + 2\mu_f)}{\alpha_n(\lambda_f + \mu_f)} \frac{I_o(\alpha_n r)}{I_o(\alpha_n)}\right) T_{3n}^{\alpha}\right] \cos \alpha_n z$$

$$+\sum_{m=1}^{MF}\left[rJ_1(\delta_m r)-\frac{\lambda_f+2\mu_f}{\lambda_f+\mu_f}\cdot\frac{2J_o(\delta_m r)}{\delta_m}\right]\left\{S_{1m}^{\delta}\sinh\delta_m z\right.$$

$$\left.+T_{1m}^{\delta}\cosh\delta_m z\right\}-\sum_{k=1}^{NSUB}\left[\frac{\sinh\beta_k z}{\cosh\beta_k l}S_{1k}+\frac{\cosh\beta_k z}{\sinh\beta_k l}T_{1k}^{\beta}\right]J_o(\beta_k r)$$

(13)

where γ_k's and β_k's are eigen values obtained from $J_1(\beta_k a)=0$, $J_o(\delta_j)=0$ and $\alpha_n=\frac{n\pi}{l}$.

Knowing u^f and w^f, the stresses can be calculated using equations (1) to (5).

2.4 Displacement Equations for Matrix

The displacement equations for u^m and w^m (radial and axial displacements for matrix) are taken as

$$u^m=\left\{E_1r^3+E_3r+E_5rz^2+F_1rz^3\right\}+\sum_{j=1}^{NF}rY_o(\delta_j r)\left\{E_{3j}^{\delta}\cosh\delta_j z\right.$$

$$\left.+F_{3j}^{\delta}\sinh\delta_j z\right\}+\sum_{k=1}^{NSUB}\left\{\frac{\cosh\beta_k z}{\cosh\beta_k l}E_{1k}^{\beta}+\frac{\sinh\beta_k z}{\sinh\beta_k l}F_{1k}^{\beta}\right\}J_1(\beta_k r)$$

$$+\sum_{j=1}^{MF}rJ_o(\delta_j r)\left\{E_{1j}^{\delta}\cosh\delta_j z+F_{1j}^{\delta}\sinh\delta_j z\right\}$$

$$+\sum_{n=1}^{NF}\left[\left\{\frac{I_1(\alpha_n r)}{I_1(\alpha_n)}E_{1n}^{\alpha}+\frac{rI_o(\alpha_n r)}{I_o(\alpha_n)}E_{3n}^{\alpha}\right\}\cosh\alpha_n z\right.$$

$$\left.+\left\{\frac{I_1(\alpha_n r)}{I_1(\alpha_n)}F_{1n}^{\alpha}+\frac{rI_o(\alpha_n r)}{I_o(\alpha_n)}F_{3n}\right\}\sin\alpha_n z\right] \qquad (14)$$

The corresponding expressionf for w^m is:

$$w^m = -\frac{4(\lambda_m + 2\mu_m)}{(\lambda_m + \mu_m)} zr^2 E_1 - \frac{\mu_m}{\lambda_m + \mu_m} zr^2 E_5$$

$$+ \frac{2}{3} z^3 \left(\frac{4\mu_m}{\lambda_m + \mu_m} E_1 - \frac{\lambda_m}{\lambda_m + \mu_m} E_5\right)$$

$$- \frac{2(\lambda_m + 2\mu_m)}{(\lambda_m + \mu_m)} r^2 z^2 F_1 + \frac{2\mu_m}{3(\lambda_m + \mu_m)} F_1 + \frac{(2\lambda_m + 3\mu_m)}{4(\lambda_m + \mu_m)} F_1 r^4$$

$$+ \sum_{m=1}^{MF} \left[rJ_1(\delta_m r) - \frac{\lambda_m + 2\mu_m}{\lambda_m + \mu_m} \cdot \frac{2J_o(\delta_m r)}{\delta_m} \right] \Big[E_{1m} \sinh \delta_m z$$

$$+ F_{1m} \cosh \delta_m z \Big] + \sum_{j=1}^{NF} \left[rY_1(\delta_j r) - \frac{\lambda_m + 2\mu_m}{\lambda_m + \mu_m} \cdot \frac{2Y_o(\delta_j r)}{\delta_j} \right]$$

$$\Big[E_{3j} \sinh \delta_j z + F_{3j} \cosh \delta_j z \Big]$$

$$- \sum_{k=1}^{NSUB} \left[\frac{\sinh \beta_k z}{\cosh \beta_k l} S_{1k} + \frac{\cosh \beta_k z}{\sinh \beta_k l} T_{1k} \right] J_o(\beta_k r)$$

$$+ \sum_{m=1}^{NF} \Big[\frac{I_o(\alpha_n r)}{I_1(\alpha_n)} E_{1n} + \Big(\frac{rI_1(\alpha_n r)}{I_o(\alpha_n)}$$

$$+ \frac{2(\lambda_m + 2\mu_m)}{n(\lambda_m + \mu_m)}) \frac{I_o(\alpha_n r)}{I_o(\alpha_n)} \Big) E_{3n} \Big] \sin \alpha_n z$$

$$- \sum_{n=1}^{NF} \Big[\frac{I_o(\alpha_n r)}{I_1(\alpha_n)} F_{1n} + \Big(\frac{rI_1(\alpha_n r)}{I_o(\alpha_n)}$$

$$+ \frac{2(\lambda_m + 2\mu_m)}{n(\lambda_m + \mu_m)}) \frac{I_o(\alpha_n r)}{I_o(\alpha_n)} F_{3n} \Big] \cos \alpha_n z \qquad (15)$$

Knowing u^m and w^m, the stresses can be calculated using equations (1) to (5).

2.5 Boundary Conditions (Exact)

For Matrix:

$$u(1, z) = 0 \tag{16}$$

$$w(1, z) = 0 \tag{17}$$

$$\sigma_z(r, 1) = 0 \tag{18}$$

$$\sigma_z(r, o) = 0 \tag{19}$$

$$\sigma_{rz}(r, 1) = 0 \tag{20}$$

$$\sigma_{rz}(r, o) = 0 \tag{21}$$

For Fiber:

$$\sigma_z(r, o) = 0 \tag{22}$$

$$\sigma_z(r, 1) = f_1 \tag{23}$$

$$\sigma_{rz}(r, o) = 0 \tag{24}$$

$$\sigma_{rz}(r, 1) = 0 \tag{25}$$

At the Interface:

(i) For bonded region ($z \leq XDB$ or $z \geq (XDB + c)$)

$$u^m(a, z) = u^f(a, z) \tag{26}$$

$$w^m(a, z) = w^f(a, z) \tag{27}$$

$$\sigma^m_{rz}(a, z) = \sigma^f_{rz}(a, z) \tag{28}$$

$$\sigma^m_z(a, z) = \sigma^f_z(a, z) \tag{29}$$

(ii) For debonded region $(XDB \leq z \leq (XDB + c))$

$$\sigma_r^m(a, z) = 0 \tag{30}$$

$$\sigma_r^f(a, z) = 0 \tag{31}$$

$$\sigma_{rz}^f(a, z) = 0 \tag{32}$$

$$\sigma_{rz}^m(a, z) = 0 \tag{33}$$

Condition for Force Equilibrium

$$\int_{z=0}^{z=1} \sigma_{rz}^m(a, z)\, dz = \frac{P}{2\pi a} \tag{34}$$

2.6 Determination of Constants in Equations (12) - (15)

The constants in the equations for u^f, w^f, u^m, w^m etc. are determined by the application of various stress and displacement boundary conditions. Truly speaking, the enforcement of proper boundary conditions would have required all summations in u^f, u^m etc. to have extended to infinity and this would have entailed the solution of an infinite matrix for constants. However, an approximate solution with a finite matrix can be obtained by truncating the infinite series.

Even with this simplified solution, one cannot satisfy the boundary conditions at the interface exactly due to the change in the conditions between bonded and unbonded regions. A collocation technique was therefore

employed to satisfy the boundary conditions in the debonded region. For convenience, these points are chosen at uniform intervals. Thus the approximate boundary conditions for debonded region become:

$$\sigma_r^m(a,\ XDB + \frac{M-1}{k-1}\ c) \quad = \quad 0 \tag{35}$$

$$\sigma_r^f(a,\ XDB + \frac{M-1}{k-1}\ c) \quad = \quad 0 \tag{36}$$

$$\sigma_{rz}^m(a,\ XDB + \frac{M-1}{k-1}\ c) \quad = \quad 0 \tag{37}$$

$$\sigma_{rz}^f(a,\ XDB + \frac{M-1}{k-1}\ c) \quad = \quad 0 \tag{38}$$

where XDB and c represent the location and extent of debonded region (Figure 2.1), k represents the number of subdivisions within the debonded region and M is an integer variable such that $1 \leq M \leq k$.

Obtaining the set of equations from the above boundary condition is fairly straightforward. What one does is to substitute the expressions for u^f, u^m etc. in the various boundary conditions. This leads to an expression either in z (for equations on z like $u^m(1,\ z) = 0$) or in r (for equations on r like $\sigma_z^f(r,\ 1) = f_1$) or in terms of constants. This expression in z (or r) is now broken down into simpler expressions and each of them is expressed as a half-range Fourier Sine series (Fourier-Bessel series for expressions on r). Regrouping the expression, one obtains a Fourier-Sine (or Fourier-Bessel) series on the left hand side. One can

follow a similar procedure on the right hand side. A term by term comparison of the series on the two sides leads to the desired set of equations. For example, consider the boundary condition $u^m(1, z) = 0$. Substitution of the expression for u^m in the above boundary condition gives:

$$u^m(1, z) = (E_1 + E_3 + E_5 z^2 + F_1 z^3)$$

$$+ \sum_{j=1}^{NF} rY_o(\delta_j) \left\{ E_{3j}^{\delta} \cosh \delta_j z + F_{3j}^{\delta} \sinh \delta_j z \right\}$$

$$+ \sum_{k=1}^{NSUB} \left\{ \frac{\cosh \beta_k z}{\cosh \beta_k l} E_{1k}^{\beta} + \frac{\sinh \beta_k z}{\sinh \beta_k l} F_{1k}^{\beta} \right\} J_1(\beta_k)$$

$$+ \sum_{j=1}^{MF} rJ_o(\delta_j) \left\{ E_{1j}^{\delta} \cosh \delta_j z + F_{1j}^{\delta} \sinh \delta_j z \right\}$$

$$+ \sum_{n=1}^{NF} \left\{ \left[E_{1n}^{\alpha} + E_{3n}^{\alpha} \right] \cos \alpha_n z + \left[F_{1n}^{\alpha} + F_{3n}^{\alpha} \right] \sin \alpha_n z \right\}$$

$$= 0$$

Replacing each function of z by its Fourier equivalent, we obtain after the regrouping of terms:

$$\sum_{m=1}^{\infty} \Big[(E_1 + E_3) X_m^1 + E_5 X_m^3 + F_1 X_m^2 + \sum_{j=1}^{NF} rY_o(\delta_j) \left\{ E_{3j}^{\delta} X_{mj}^7 + F_{3j}^{\delta} X_{mj}^8 \right\}$$

$$+ \sum_{n=1}^{NF} \left\{ \left[E_{1n}^{\alpha} + E_{3n}^{\alpha} \right] X_{mn}^6 + \left[F_{1n}^{\alpha} + F_{3n}^{\alpha} \right] \delta_n^m \right\}$$

$$+ \sum_{k=1}^{NSUB} \left\{ \frac{\cosh \beta_k z}{\cosh \beta_k l} E_{1k}^{\beta} + \frac{\sinh \beta_k z}{\sinh \beta_k l} F_{1k}^{\beta} \right\} J_1(\beta_k) \Big] \sin \alpha_m z = 0$$

Thus equating each term of the Fourier Sine series to zero, we get:

$$(E_1 + E_3)X^1_m + E_5X^3_m + F_1X^2_m + \sum_{j=1}^{NF} rY_o(\delta_j)\left\{E^\delta_{3j}X^7_{mj} + F^\delta_{3j}X^8_{mj}\right\}$$

$$+ \sum_{n=1}^{NF}\left[\left\{E^\alpha_{1n} + E^\alpha_{3n}\right\}X^6_{mn} + \left\{F^\alpha_{1n} + F^\alpha_{3n}\right\}\delta^m_n\right]$$

$$+ \sum_{k=1}^{NSUB}\left[\frac{X^9_{mk}}{\cosh \beta_k l}E_{1k} + \frac{X^{10}_{mk}}{\sinh \beta_k l}F_{1k}\right]J_1(\beta_k) = 0$$

Similarly equations can be obtained for the remaining boundary conditions. For clarity sake, the various terms have been grouped under different categories. They are:

(SKNWN) = Column vector containing the constants S_1, S_3, S_5, T_1, T_5 and the column vectors S^α_{3j}, T^α_{3j}, S^δ_{1j}, T^δ_{1j}.

Thus

$$(SKNWN) = \left\{S_1, S_3, S_5, T_1, T_5, S^\alpha_{31}..S^\alpha_{3m}, T^\alpha_{31}..T^\alpha_{3m}, S^\delta_{11}..S^\delta_{1m}, T^\delta_{11}..T^\delta_{1m}\right\}$$

$$= \left\{S_1, S_3, S_5, T_1, T_5, (S^\delta_{3j})^T, (T^\delta_{3j})^T, (S^\delta_{1j})^T, (T^\delta_{1j})^T\right\}^T$$

The other vectors are:

$$\underset{(4M_f+1)}{(EKNWN)} = \left\{\left[E_1, E_3, E_5, F_1, (E^\alpha_{3j})^T, (F^\alpha_{3j})^T, (E^\delta_{1j})^T, (F^\delta_{1j})^T, (E^\delta_{3j})^T, (F^\delta_{3j})^T\right]^T\right\}$$

$$\underset{(2N_f)}{(\text{SALPH})} = \left\{ (S^{\alpha}_{1j})^T,\ (T^{\alpha}_{1j})^T \right\}^T$$

$$\underset{(2N_f)}{(\text{EALPH})} = \left\{ (E^{\alpha}_{1j})^T,\ (F^{\alpha}_{1j})^T \right\}^T$$

$$\underset{(2k)}{(\text{SBETA})} = \left\{ (S^{\beta}_{1j})^T,\ (T^{\beta}_{1j})^T \right\}^T$$

$$\underset{(2k)}{(\text{EBETA})} = \left\{ (E^{\beta}_{1j})^T,\ (F^{\beta}_{1j})^T \right\}^T$$

$$\underset{(4M_f+1)}{(\text{SCONST})} = \text{Load vector associated with fibers}$$

$$= \left\{ (0)^T,\ (\text{PHI}*G^1_m)^T,\ (0)^T,\ (0)^T,\ (\text{P1}*G^1_m)^T \right\}^T$$

Thus the equation $u^m(1,z) = 0$ can be written in the matrix form as:

$$\left[X^1_m \quad X^1_m \quad X^3_m \quad X^2_m \quad X^6_{mn} \quad \delta^m_n \quad (0)^T \quad (0)^T \quad X^7_{mn} \quad X^8_{mn} \right] *$$

$$\left\{ E_1,\ E_3,\ E_5,\ F_1,\ (E^{\alpha}_{3j})^T,\ (F^{\alpha}_{3j})^T,\ (E^{\beta}_{1j})^T,\ (F^{\beta}_{1j})^T,\ (E^{\beta}_{3j})^T,\ (F^{\beta}_{3j})^T \right\}^T$$

$$+ \left[X^6_{mn} \quad \delta^m_n \right] \left[(S^{\alpha}_{1j})^T,\ (T^{\alpha}_{1j})^T \right]^T$$

$$+ \left[\left(\frac{J_1(\beta_j)X^9_{mj}}{\cosh \beta_j l}\right) \quad \left(\frac{J_1(\beta_j)X^{10}_{mj}}{\sinh \beta_j l}\right) \right] \left\{ (S^{\beta}_{1j})^T,\ (T^{\beta}_{1j})^T \right\} = 0$$

or

$$(B'_M)\ (\text{EKNWN}) + (B''_M)\ (\text{EALPH}) + (B'''_M)\ (\text{EBETA}) = (0)$$

where $(B'_M) = \left[X^1_m \quad X^1_m \quad X^3_m \quad X^2_m \quad X^6_{mn} \quad \delta^m_n \quad 0^T \quad 0^T \quad X^7_{mn} \quad X^8_{mn} \right]$

$$(B_M'') = (X^6_{mn} \; \delta_{mn}),$$

$$(B_M''') = \left(\frac{X^9_{mj} e_1}{\cosh \beta_j l} \quad \frac{X^{10}_{mj} e_1}{\sinh \beta_j l}\right). \qquad \text{where } e_1 = (J_1(\beta_j))$$

The equations to be satisfied on the fiber, matrix and the interface are written in a form similar to the above as shown in the Appendix II.

Solution of Equations:

The equations can be written in the following form.

(i) Equations on fiber:

$$\begin{array}{ccccccc} (A_F^1) & (SKNWN) = & (A_F^2) & (SALPH) + & (A_F^3) & (SBETA) + & (SCONST) \\ (4M_f+1)\text{x}(4M_f+1) & & (4M_f+1)\text{x}2N_f & & (4M_f+1)\text{x}2k & & (4M_f+1) \end{array} \qquad \text{(i)}$$

(ii) Equations on matrix:

$$\begin{array}{cccccc} (B_M^1) & (EKNWN) = & (B_M^2) & (EALPH) + & (B_M^3) & (EBETA) \\ (4M_f+2N_f)\text{x}(4M_f+2N_f) & & (4M_f+2N_f)\text{x}2N_f & & (4M_f+2N_f)2k & \end{array} \qquad \text{(ii)}$$

(iii) Equation on interface (Bonded region):

$$\begin{array}{cccccc} (A_F^4) & (SKNWN) + & (A_F^5) & (SALPH) + & (A_F^6) & (SBETA) \\ 4N_f\text{x}(4M_f+1) & & 4N_f\text{x}2N_f & & 4N_f\text{x}2k & \end{array}$$

$$\begin{array}{ccccccc} = & (B_M^4) & (EKNWN) + & (B_M^5) & (EALPH) + & (B_M^6) & (EBETA) \\ \text{order} \rightarrow & 4N_f\text{x}(4M_f+2N_f) & & 4N_f\text{x}2N_f & & 4N_f\text{x}2k & \end{array} \qquad \text{(iii)}$$

(iv) Equations on interface (Debonded region):

$$\underset{2k\times(4M_f+1)}{(A_F^7)}\ (SKNWN) + \underset{(2k\times 2N_f)}{(A_F^8)}\ (SALPH) + \underset{(2k\times 2k)}{(A_F^9)}\ (SBETA) = (O) \qquad \text{(iv)}$$

$$\underset{2k\times(4M_f+2N_f)}{(B_M^7)}\ (EKNWN) + \underset{2k\times 2N_f}{(B_M^8)}\ (EALPH) + \underset{2k\times 2k}{(B_M^9)}\ (EBETA) = O \qquad \text{(v)}$$

From equations (i) and (ii)

$$(SKNWN) = (A_F^1)^{-1}\ (A_F^2)(SALPH) + (A_F^1)^{-1}\ (A_F^3)(SBETA) + (A_F^1)^{-1}(SCONST) \qquad \text{(vi)}$$

$$(EKNWN) = (B_M^1)^{-1}\ (B_M^2)(EALPH) + (B_M^1)^{-1}\ (B_M^3)(EBETA) \qquad \text{(vii)}$$

Substituting the values in (iii), (iv) and (v), one gets after rearrangement:

$$\underset{(4N_f+4k)\times(4N_f+4k)}{\begin{bmatrix} \underset{4N_f\times 2N_f}{(AF412+A_F^5)} & \underset{4N_f\times 2N_f}{-(BM412+B_M^5)} & \underset{4N_f\times 2k}{(AF413+A_F^6)} & \underset{4N_f\times 2k}{-(BM413+B_M^6)} \\ \underset{2k\times 2N_f}{(AF712+A_F^8)} & \underset{2k\times 2N_f}{(O)} & \underset{2k\times 2k}{(AF713+A_F^9)} & \underset{2k\times 2k}{(O)} \\ \underset{2k\times 2N_f}{(O)} & \underset{2k\times 2N_f}{(BM712+B_M^8)} & \underset{2k\times 2k}{(O)} & \underset{2k\times 2k}{(BM713+B_M^9)} \end{bmatrix}} \underset{(4N_f+4k)}{\begin{Bmatrix} (SALPH) \\ (EALPH) \\ (SBETA) \\ (EBETA) \end{Bmatrix}} = \underset{(4N_f+4k)}{\begin{Bmatrix} \underset{(4N_f)}{-(AF41)x\ (SCONST)} \\ \underset{(2k)}{-(AF71)x\ (SCONST)} \\ \underset{(2k)}{(O)} \end{Bmatrix}} \qquad \text{(viii)}$$

where

$$(AF412) = (A_F^4)\ (A_F^1)^{-1}\ (A_F^2)$$

$$(BM412) = (B_M^4)\ (B_M^1)^{-1}\ (B_M^2)$$

$$(AF413) = (A_F^4)\ (A_F^1)^{-1}\ (A_F^3)$$

$$(BM413) = (B_M^4)\ (B_M^1)^{-1}\ (B_M^3)$$

$$(AF712) = (A_F^7)\ (A_F^1)^{-1}\ (A_F^2)$$

$$(BM712) = (B_M^7)\ (B_M^1)^{-1}\ (B_M^2)$$

$$(AF713) = (A_F^7)\ (A_F^1)^{-1}\ (A_F^3)$$

$$(BM713) = (B_M^7)\ (B_M^1)^{-1}\ (B_M^3)$$

$$(AF41) = (A_F^4)\ (A_F^1)^{-1}$$

$$(AF71) = (A_F^7)\ (A_F^1)^{-1}$$

The equation (viii) is a set of $(4N_f+4k)$ equations in $(4N_f+4k)$ unknowns. Thus it can be solved for (SALPH), (EALPH), (SBETA) and (EBETA). Then (SKNWN) and (EKNWN) can be determined using (vi) and (vii). Knowing all these constants, the values of u^f, u^m, etc. can be found by simple substitution in Eqs. (12) to (15).

3. DISCUSSION & CONCLUSIONS

3.1 Numerical Computations

The following data are used to get the solutions for a hypothetical case:

Poisson's Ratio for Fiber = 0.25

Poisson's Ratio for Matrix = 0.27

Ratio of Elastic Modulii of Fiber to Matrix = 20

Non-dimensional Length = 2.0

Non-dimensional Radius of Case (Matrix) = 1.0

Non-dimensional Radius of Case (Fiber) = 0.4

Applied Load (Non-dimensionalized)

= Ratio of Load Applied at the Fiber End to the Shear Modulus of Matrix

= 0.00001

Non-dimensional Location of Debonded Region (XDB) = 1.2

Non-dimensional Extent of Debonded Region (C) = 0.5

For convenience, the computations were carried out in two stages. In the first stage, the coefficients of Fourier-Bessel series were evaluated using exact formulae wherever they were available and resorting to numerical integration (Simpson's rule with 100 interval) elsewhere. The results so obtained were used as input for further computations.

In the second stage, the infinite set of equations (Appendix II) are solved by considering only three terms in the solution. The LU-decomposition method is then employed to obtain the variables of the equations. The displacement field is then obtained by using equations (12) to (15). Stresses can be computed by using equations (2) to (5). Appendix III gives the flow chart for computational work which has been used to develop the programme for DEC 1090.

3.2 Results

Figure 3.1 shows the plot of axial stress with length. It is observed that σ_z attains a maximum at the middle of the length and goes to zero at the two ends.

Figure 3.2 depicts the variation of shear stress with length. The shear stress σ_{rz} is zero at z = 0 and reaches a maximum value at z = 1/4. It tends to zero at z = 1/2 and remains constant beyond this value.

The variation of radial displacement with length is shown in Figure 3.3. It is observed that the displacement increases continuously till z = 1/2. It then keeps falling down till the end of the debonded region and then remains nearly constant.

Variation of the axial displacement with length is shown in Figure 3.4. It is observed that the axial displacement increases monotonically with increasing z.

Figure 3.5 is drawn to show the variation of σ_r with length. The radial stress reaches a maximum in the bonded region and becomes zero as expected in the unbonded zone.

3.3 Conclusions

Following conclusions can be drawn on the basis of this limited study.

1. The axial stresses are of a much higher order of magnitude than the shear and radial stresses, which implies that the load transfer is mainly through axial stresses.

2. From Figures 3.2 and 3.5 it can be seen that the stresses σ_{rz} and σ_r reach their maximum values in the upper region of the length ($0 \leq z \leq l/2$) and hence there is a liklihood of crack initiating in this region, rather than in the lower portion.

3.4 Suggestions for Future Work

1. It is suggested that this analysis be carried out by taking a few more terms in the infinite series to study the convergence of the solution and also give a better picture regarding the variation of stresses and displacement along z.

2. The model can be improved by considering multiple fibers in a given matrix which may result in redistribution of stresses in the debonded zone.

LIST OF REFERENCES

[1] Agarwal, B.D. Lecture Notes on Composite Materials. Department of Mechanical Engineering, I.I.T. Kanpur.

[2] Cooper, G.A. & Kelly, A. (1968) Role of the Interface in the Fracture of Fiber-composite Materials in 'Interfaces in Composites'. STP 452, pp. 90-106, Amer. Soc. Test. Mater., Philadelphia, Pennsylvania.

[3] Ebert, L.J. & Gadd, J.D. Fiber Composite Materials. Chap. 5, A.S.M. Publication.

[4] Chamis & Sendeckyj. Critique on Theories Predicting Thermo-elastic Properties of Fibrous Composites. J. Composite Materials, Vol. 2, No. 3 (July 1968), p. 332.

[5] Holister, G.S. & Thomas, C. Fiber Reinforced Materials. Elsevier Publishing Co. Ltd., N.Y., 1966.

[6] Metcalfe, A.G. Interfaces in Metal Matrix Composites. Vol. 1 in Composite Materials, ed. Broutman & Krock, Academic Press, N.Y. (1974).

[7] Lawrence, P. Some Theoretical Considerations of Fiber Pull-out from an Elastic Matrix. J. of Material Science, Vol. 7 (1972) Jan., p. 1.

[8] Murthy, P.N. Analysis of Finite Length, Circular, Encased Viscoelastic Cylinder. Ph.D. Dissertation (1962), University of Illinois at Urbana Champagne.

[9] Sarma, A.V.S.R.J. Interfacial Shear Distribution in GFRP. M.Tech. Thesis, Department of Aeronautical Engineering, I.I.T. Kanpur, 1976.

APPENDIX I: EXPANSIONS

Expansion	Remarks
$1 = \sum_{m=1}^{\infty} G_m^1 J_o(\delta_m r)$	$G_m^1 = 2/(\delta_m J_1(\delta_m))$
$r = \sum_{m=1}^{\infty} G_m^2 J_o(\delta_m r)$	$G_m^2 = 2/J_1^2(\delta_m) \int_o^1 r^2 J_o(\delta_m r)dr$
$r^2 = \sum_{m=1}^{\infty} G_m^3 J_o(\delta_m r)$	$G_m^3 = 2(\delta_m^2 - 4)/(\delta_m^3 J_1(\delta_m))$
$r^3 = \sum_{m=1}^{\infty} G_m^4 J_o(\delta_m r)$	$G_m^4 = 2/J_1^2(\delta_m) \int_o^1 r^4 J_o(\delta_m r)dr$
$I_o(\alpha_n r) = \sum_{m=1}^{\infty} G_{mn}^5 J_o(\delta_m r)$	$G_{mn}^5 = 2I_o(\alpha_n)\delta_m/(\delta_m^2 + \alpha_n^2)J_1(\delta_m)$
$I_1(\alpha_n r) = \sum_{m=1}^{\infty} G_{mn}^6 J_o(\delta_m r)$	$G_{mn}^6 = 2/J_1^2(\delta_m) \int_o^1 rI_1(\alpha_n r)J_o(\delta_m r)dr$
$rI_o(\alpha_n r) = \sum_{m=1}^{\infty} G_{mn}^7 J_o(\delta_m r)$	$G_{mn}^7 = 2/J_1^2(\delta_m) \int_o^1 r^2 I_o(\alpha_n r)J_o(\delta_m r)dr$
$rI_1(\alpha_n r) = \sum_{m=1}^{\infty} G_{mn}^8 J_o(\delta_m r)$	$G_{mn}^8 = 2/J_1^2(\delta_m) \int_o^1 r^2 I_1(\alpha_n r)J_o(\delta_m r)dr$
$I_1(\alpha_n r)/r = \sum_{m=1}^{\infty} G_{mn}^9 J_o(\delta_m r)$	$G_{mn}^9 = 2/J_1^2(\delta_m) \int_o^1 I_1(\alpha_n r)J_o(\delta_m r)dr$
$J_o(\beta_n r) = \sum_{m=1}^{\infty} G_{mn}^{10} J_o(\delta_m r)$	$G_{mn}^{10} = 2J_o(\beta_n)\delta_m/(\beta_n^2 - \delta_m^2)J_1(\delta_m)$
$J_1(\beta_n r) = \sum_{m=1}^{\infty} G_{mn}^{11} J_o(\delta_m r)$	$G_{mn}^{11} = 2/J_1^2(\delta_m) \int_o^1 rJ_1(\beta_n r)J_o(\delta_m r)dr$
$J_1(\beta_n r)/r = \sum_{m=1}^{\infty} G_{mn}^{12} J_o(\delta_m r)$	$G_{mn}^{12} = 2/J_1^2(\delta_m) \int_o^1 J_1(\beta_n r)J_o(\delta_m r)dr$

$$rJ_o(\delta_n r) = \sum_{m=1}^{\infty} G^{13}_{mn} J_o(\delta_m r) \qquad G^{13}_{mn} = 2/J_1^2(\delta_m) \int_0^1 r^2 J_o(\delta_n r) J_o(\delta_m r)dr$$

$$rJ_1(\delta_n r) = \sum_{m=1}^{\infty} G^{14}_{mn} J_o(\delta_m r) \qquad G^{14}_{mn} = 2/J_1^2(\delta_m) \int_0^1 r^2 J_1(\delta_n r) J_o(\delta_m r)dr$$

$$J_1(\delta_n r) = \sum_{m=1}^{\infty} G^{15}_{mn} J_o(\delta_m r) \qquad G^{15}_{mn} = 2/J_1^2(\delta_m) \int_0^1 rJ_1(\delta_n r) J_o(\delta_m r)dr$$

$$rY_o(\delta_n r) = \sum_{m=1}^{\infty} G^{16}_{mn} J_o(\delta_m r) \qquad G^{16}_{mn} = 2/J_1^2(\delta_m) \int_0^1 r^2 Y_o(\delta_n r) J_o(\delta_m r)dr$$

$$rY_1(\delta_n r) = \sum_{m=1}^{\infty} G^{17}_{mn} J_o(\delta_m r) \qquad G^{17}_{mn} = 2/J_1^2(\delta_m) \int_0^1 r^2 Y_1(\delta_n r) J_o(\delta_m r)dr$$

$$Y_1(\delta_n r) = \sum_{m=1}^{\infty} G^{18}_{mn} J_o(\delta_m r) \qquad G^{18}_{mn} = 2/J_1^2(\delta_m) \int_0^1 rY_1(\delta_n r) J_o(\delta_m r)dr$$

$$Y_o(\delta_n r) = \sum_{m=1}^{\infty} G^{19}_{mn} J_o(\delta_n r) \qquad G^{19}_{mn} = 2/J_1^2(\delta_m) \int_0^1 rY_o(\delta_n r) J_o(\delta_m r)dr$$

$$J_o(\delta_n r) = \sum_{m=1}^{\infty} \Delta^n_m J_o(\delta_m r) \qquad \Delta^n_m = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}$$

$$1 = \sum_{n=1}^{\infty} X^1_n \sin \lambda_n z \qquad X^1_n = 2(1 - (-1)^n)/\pi n$$

$$z = \sum_{n=1}^{\infty} X^2_n \sin \lambda_n z \qquad X^2_n = -2l/(n\pi)(-1)^n$$

$$z^2 = \sum_{n=1}^{\infty} X^3_n \sin \lambda_n z \qquad X^3_n = -2l^2/(n\pi) - 2l^2/n^3\pi^3(1-(-1)^n)$$

$$z^3 = \sum_{n=1}^{\infty} X^4_n \sin \lambda_n z \qquad X^4_n = -2l^3(-1)^n/n\pi + 6/n^3\pi^3(-1)^n$$

$$z^4 = \sum_{n=1}^{\infty} X^5_n \sin \lambda_n z \qquad X^5_n = -2l^4\Big[(-1)^n/n\pi + 12/n^3\pi^3 + 24/n^5\pi^5(1-(-1)^n)\Big]$$

$$\cos \alpha_j z = \sum X^6_{nj} \sin \alpha_n z \qquad X^6_{nj} = \begin{cases} 0 & \text{if } (n-j) \text{ is even} \\ 4n/(n^2-j^2)\pi & \text{if } (n-j) \text{ is odd} \end{cases}$$

$$\cosh \delta_j z = \sum X^7_{nj} \sin \alpha_n z \qquad X^7_{nj} = \frac{2n\pi}{(n^2\pi^2 + \delta^2_j l^2)} (1+(-1)^n \cosh \delta_j l)$$

$$\sinh \delta_j z = \sum X^8_{nj} \sin \alpha_n z \qquad X^8_{nj} = -\frac{2n\pi}{(n^2\pi^2 + \delta^2_j l^2)} (-1)^n$$

$$\cosh(\beta_j z) = \sum X^9_{nj} \sin \alpha_n z \qquad X^9_{nj} = \frac{2n\pi}{(n^2\pi^2 + \beta^2_j l^2)} (1+(-1)^n \cosh \beta_j l)$$

$$\sinh(\beta_j z) = \sum X^{10}_{nj} \sin \alpha_n z \qquad X^{10}_{nj} = -\frac{2n\pi}{(n^2\pi^2 + \beta^2_j l^2)} (-1)^n$$

APPENDIX II: DETAILS OF EQUATIONS APPLYING BOUNDARY CONDITIONS

(i) Equations on Fiber

$$\Big[\mu_F\left(\frac{-4(3\lambda_f+4)}{(\lambda_f+1)}\,G_m^3\right) \quad \mu_F(2\lambda_f G_m^1) \quad \mu_F\left(\frac{-(\lambda_f+2)}{(\lambda_f+1)}\,G_m^3\right) \quad (0) \quad (0)$$

$$-\,2\mu_F\alpha_j\left(\frac{(3\lambda_f+4)}{(\lambda_f+1)\alpha_j}\,\frac{G_{mj}^5}{I_o(\alpha_j)} + \frac{G_{mj}^8}{I_o(\alpha_j)}\right) \quad (0) \quad 2\mu_F\beta_j\left(\frac{(3\lambda_f+4)}{(\lambda_f+1)}\cdot\frac{\lambda_j^m}{-(\beta_j)}\right.$$

$$\left.+\,G_{mj}^{14}\right) \quad (0)\Big] \quad (\text{SKNWN}) \quad = \quad \Big[\left(2\mu_F\alpha_j\left(\frac{G_{mj}^5}{I_1(\alpha_j)}\right) \quad (0)\right] \quad (\text{SALPH})$$

$$+\;\Big[\left(2\mu_F\beta_j\left(\frac{G_{mj}^{10}}{\cosh\beta_j l}\right)\right) \quad (0)\Big] \quad (\text{SBETA}) \qquad (\text{A.22})$$

$$\Big[\mu_F\left(\frac{8(\lambda_f+2)}{(\lambda_f+1)}\,l^2G_m^1 - \frac{4(3\lambda_f+4)}{(\lambda_f+1)}\,G_m^3\right) \quad \mu_F(2\lambda_f G_m^1) \quad \mu_F\left(-\frac{(\lambda_f+2)}{(\lambda_f+1)}\,G_m^3\right.$$

$$\left.-\,\frac{2\lambda_f}{(\lambda_f+1)}\,l^2G_m^1\right) \quad \mu_F\left(-\frac{4(3\lambda_f+4)}{(\lambda_f+1)}\,lG_m^3 + \frac{8(\lambda_f+2)}{3(\lambda_f+1)}\,l^3G_m^1\right.$$

$$\mu_F\left(-\frac{3(\lambda_f+2)}{(\lambda_f+1)}\,lG_m^3 - \frac{2\lambda_f}{(\lambda_f+1)}\,l^3G_m^1\right) \quad \left(-2\mu_F\alpha_j(-1)^j\left(\frac{(3\lambda_f+4)}{(\lambda_f+1)\alpha_j}\,\frac{G_{mj}^5}{I_o(\alpha_j)}\right.\right.$$

$$\left.+\,\frac{G_{mj}^8}{I_o(\alpha_j)}\right) \quad (0) \quad 2\mu_F\cosh\beta_j l\left(-\frac{(3\lambda_f+4)}{(\lambda_f+1)}\,\frac{\lambda_j^m}{\beta_j} + G_{mj}^{14}\right)$$

$$2\mu_F\beta_j\sinh\beta_j l\left(-\frac{(3\lambda_f+4)}{(\lambda_f+1)}\,\frac{\lambda_j^m}{\beta_j} + G_{mj}^{14}\right)\Big] \quad (\text{SKNWN}) \quad =$$

$$\Big[2\mu_F\alpha_j(-1)^j\left(\frac{G_{mj}^5}{I_1(\alpha_j)}\right) \quad (0)\Big] \quad (\text{SALPH}) + \Big[2\mu_F\beta_j(G_{mj}^{10})$$

$$2\mu_F\beta_j(G_{mj}^{10})\Big] \quad (\text{SBETA}) + \Big[(\text{PHI} \times G_m^1)\Big] \qquad (\text{A.23})$$

$$\Big[(0) \quad (0) \quad (0) \quad \zeta_F\Big(\frac{(3\lambda_f+4)}{(\lambda_f+1)} G_m^4\Big) \quad \zeta_F\Big(\frac{3(\lambda_f+2)}{4(\lambda_f+1)} G_m^4\Big) \quad (0)$$

$$2\zeta_F\alpha_j\Big(\frac{(\lambda_f+2)}{(\lambda_f+1)} \frac{G_{mj}^6}{\alpha_j I_o(\alpha_j)} + \frac{G_{mj}^7}{I_o(\alpha_j)}\Big) \quad (0) \quad 2\zeta_F\beta_j\Big(\frac{(\lambda_f+2)}{(\lambda_f+1)} \frac{G_{mj}^{15}}{\beta_j} + G_{mj}^{13}\Big)\Big]$$

$$(SKNWN) = \Big[(0) \quad 2\zeta_F\alpha_j\Big(-\frac{G_{mj}^6}{I_1(\alpha_j)}\Big)\Big] \; (SALPH) + \Big[(0)$$

$$2\zeta_F\beta_j\Big(-\frac{G_{mj}^{11}}{\sinh \beta_j l}\Big)\Big] \; (SBETA) \qquad (A.24)$$

$$\Big[\zeta_F\Big(-\frac{8(\lambda_f+2)}{(\lambda_f+1)} lG_m^2\Big) \quad (0) \quad \zeta_F\Big(\frac{2\lambda_f}{(\lambda_f+1)} lG_m^2\Big) \quad \zeta_F\Big(\frac{(3\lambda_f+4)}{(\lambda_f+1)} G_m^4$$

$$-\frac{4(\lambda_f+2)}{(\lambda_f+1)} l^2G_m^2\Big) \quad \zeta_F\Big(\frac{3(\lambda_f+2)}{4(\lambda_f+1)} G_m^4 + \frac{3\lambda_f}{(\lambda_f+1)} l^2G_m^2\Big) \quad (0)$$

$$2\zeta_F\alpha_j(-1)^j\Big(\frac{(\lambda_f+2)}{(\lambda_f+1)} \frac{G_{mj}^6}{\alpha_j I_o(\alpha_j)} + \frac{G_{mj}^7}{I_o(\alpha_j)}\Big)$$

$$2\zeta_F\beta_j \sinh \beta_j l\Big(\frac{(\lambda_f+2)}{(\lambda_f+1)} \frac{G_{mj}^{15}}{\beta_j} + G_{mj}^{13}\Big) \quad 2\zeta_F\beta_j \cosh \beta_j l\Big(\frac{(\lambda_f+2)}{(\lambda_f+1)} \frac{G_{mj}^{15}}{\beta_j}$$

$$+ G_{mj}^{13}\Big)\Big] \; (SKNWN) = \Big[(0) \quad 2\zeta_F\alpha_j(-1)^j\Big(-\frac{G_{mj}^6}{I_1(\alpha_j)}\Big)\Big] \; (SALPH)$$

$$+ \Big[2\zeta_F\beta_j \tanh \beta_j l(-G_{mj}^{11}) \quad 2\zeta_F\beta_j\Big(-\frac{G_{mj}^{11}}{\tanh \beta_j l}\Big)\Big] \; (SBETA)$$

(A.25)

$$\Big[\ \zeta_F\Big(-\frac{8(\lambda_f+1)}{(\lambda_f+1)}\frac{l^2a}{2}\Big)\quad (0)\quad \zeta_F\Big(\frac{\lambda_f}{(\lambda_f+1)}\ l^2a\quad {}_F\Big(\frac{(3\lambda_f+4)}{(\lambda_f+1)}\ l^3a$$

$$-\frac{4(\lambda_f+2)}{(\lambda_f+1)}\frac{al^3}{3}\Big)\quad \zeta_F\Big(\frac{3(\lambda_f+2)}{4(\lambda_f+1)}\ la^3+\frac{3\lambda_f}{(\lambda_f+1)}\frac{al^3}{3}\Big)$$

$$2\zeta_F((-1)^j-1)\Big(\frac{(\lambda_f+2)}{(\lambda_f+1)}\frac{I_1(\alpha_j a)}{\alpha_j I_o(\alpha_j)}+\frac{aI_o(\alpha_j a)}{I_o(\alpha_j)}\Big)\quad (0)$$

$$2\zeta_F\cosh\beta_j l\Big(\frac{(\lambda_f+2)}{(\lambda_f+1)}\frac{J_1(\beta_j a)}{\beta_j}+aJ_o(\beta_j a)\Big)$$

$$2\zeta_F\sinh\beta_j l\Big(\frac{(\lambda_f+2)}{(\lambda_f+1)}\frac{J_1(\beta_j a)}{\beta_j}+aJ_o(\beta_j a)\Big)\Big]\quad (\mathrm{SKNWN})\quad =$$

$$\Big[\ 2\zeta_F(1-(-1)^j)\Big(\frac{I_1(\alpha_j a)}{I_1(\alpha_j)}\Big)\quad (0)\ \Big]\quad (\mathrm{SALPH})$$

$$+\Big[\ 2\zeta_F J_1(\beta_j a)\ \Big(\frac{1-\cosh\beta_j l}{\cosh\beta_j l}\Big)\quad -2\zeta_F J_1(\beta_j a)(-1)\Big)\Big]\quad (\mathrm{SBETA})$$

$$+\Big[\ P1*G_m^1\ \Big] \tag{A.34}$$

(ii) Equations on Matrix

$$\Big[(X_m^1)\quad (X_m^1)\quad (X_m^3)\quad (X_m^2)\quad (X_{mj}^6)\quad (\lambda_j^m)\quad (0)\quad (0)\quad (Y_o(\delta_j))X_{mj}^7$$

$$(Y_o(\delta_j))X_{mj}^8\Big]\ (\mathrm{EKNWN})\quad =\Big[(-X_{mj}^6)\quad (-\Delta_j^m)\Big]\ (\mathrm{EALPH})\quad +$$

$$\Big[\Big(\frac{J_1(\beta_j)}{\cosh\beta_j l}\Big)(-X_{mj}^9)\quad \Big(\frac{J_1(\beta_j)}{\sinh\beta_j l}\Big)(-X_{mj}^{10})\Big]\quad (\mathrm{EBETA}) \tag{A.16}$$

$$\Big[\Big(-\frac{(\lambda_m+2)4}{(\lambda_m+1)}\, X_m^2 + \frac{8}{3(\lambda_m+1)}\, X_m^4\Big) \qquad (0) \qquad \Big(-\frac{1}{(\lambda_m+1)}\, X_m^2 - \frac{2\lambda_m}{3(\lambda_m+1)}\, X_m^4\Big)$$

$$\Big(-\frac{2(\lambda_m+2)}{(\lambda_m+1)}\, X_m^3 + \frac{2}{3(\lambda_m+1)}\, X_m^5 + \frac{(2\lambda_m+3)}{4(\lambda_m+1)}\, X_m^1\Big)$$

$$-\Big(\frac{I_1(\alpha_j)}{I_o(\alpha_j)} + \frac{2(\lambda_m+2)}{(\lambda_m+1)\;\;j}\Big)\;\Delta_j^m \qquad \Big(\frac{I_1(\alpha_j)}{I_o(\alpha_j)} + \frac{2(\lambda_m+2)}{(\lambda_m+1)\;\;j}\Big)\, X_{mj}^6$$

$$\Big(J_1(\delta_j) - \frac{(\lambda_m+2)}{(\lambda_m+1)}\cdot\frac{2J_o(\delta_j)}{\delta_j}\Big)\, X_{mj}^8 \qquad \Big(J_1(\delta_j) - \frac{(\lambda_m+2)}{(\lambda_m+1)}\cdot\frac{2J_o(\delta_j)}{\delta_j}\Big)\, X_{mj}^7$$

$$\Big(Y_1(\delta_j) - \frac{(\lambda_m+2)}{(\lambda_m+1)}\cdot\frac{2Y_o(\delta_j)}{\delta_j}\Big)\, X_{mj}^8 \qquad \Big(Y_1(\delta_j) - \frac{(\lambda_m+2)}{(\lambda_m+1)}\cdot\frac{2Y_o(\delta_j)}{\delta_j}\Big)\, X_{mj}^7\Big]$$

$$(\mathrm{EKNWN}) \;=\; \Big[\Big(\frac{I_o(\alpha_j)}{I_1(\alpha_j)}\Big)\;\Delta_j^m \qquad \Big(-\frac{I_o(\alpha_j)}{I_1(\alpha_j)}\Big)\, X_{mj}^6\Big]\;(\mathrm{EALPH})$$

$$+\Big[\Big(\frac{J_o(\beta_j)}{\cosh \beta_j l}\Big)\, X_{mj}^{10} \qquad \Big(\frac{J_o(\beta_j)}{\sinh \beta_j l}\Big)\, X_{mj}^9\Big]\;(\mathrm{EBETA}) \qquad\qquad (A.17)$$

$$\Big[\Big(-\frac{4(3\lambda_m+4)}{(\lambda_m+1)}\, G_m^3\Big) \qquad (2\lambda_m G_m^1) \qquad \Big(-\frac{(\lambda_m+2)}{(\lambda_m+1)}\, G_m^3\Big) \qquad (0)$$

$$\Big((-2\alpha_j)\mathrm{x}\Big(\frac{3\lambda_m+4)}{(\lambda_m+1)\alpha_j}\cdot\frac{G_{mj}^5}{I_o(\alpha_j)} + \frac{G_{mj}^8}{I_o(\alpha_j)}\Big)\Big) \qquad (0)$$

$$(2\delta_j)\mathrm{x}\Big(-\frac{(3\lambda_m+1)}{(\lambda_m+1)}\cdot\frac{\Delta_j^m}{\delta_j} + G_{mj}^{14}\Big) \qquad (0) \qquad (2\delta_j)\mathrm{x}\Big(-\frac{(3\lambda_m+4)}{(\lambda_m+1)}\cdot\frac{G_{mj}^{19}}{j} + G_{mj}^{17}\Big)$$

$$(0)\Big]\;(\mathrm{EKNWN}) \;=\; \Big[(2\alpha_j)\mathrm{x}\Big(\frac{G_{mj}^5}{I_1(\alpha_j)}\Big) \qquad (0)\Big]\;(\mathrm{EALPH}) \;+$$

$$\Big[(2\beta_j)\mathrm{x}\Big(\frac{G_{mj}^{10}}{\cosh \beta_j l}\Big) \qquad (0)\Big]\;(\mathrm{EBETA}) \qquad\qquad (A.19)$$

$$\Big[\Big(\frac{8(\lambda_m+2)}{(\lambda_m+1)}\ l^2G_m^1 - \frac{4(3\lambda_m+4)}{(\lambda_m+1)}\ G_m^3\Big) \quad (2\lambda_m G_m^1) \quad \Big(-\frac{(\lambda_m+2)}{(\lambda_m+1)}\ G_m^3 -$$

$$\frac{2\lambda_m}{(\lambda_m+1)}\ l^2G_m^1\Big) \quad \Big(-\frac{4(3\lambda_m+4)}{(\lambda_m+1)}\ lG_m^3 + \frac{8(\lambda_m+2)}{3(\lambda_m+1)}\ l^2G_m^1\Big)$$

$$(-2\alpha_j(-1)^j)x\Big(\frac{(3\lambda_m+4)}{(\lambda_m+1)\alpha_j}\cdot\frac{G_{mj}^5}{I_o(\alpha_j)} + \frac{G_{mj}^8}{I_o(\alpha_j)}\Big) \quad (0)$$

$$(2\delta_j\ \cosh\ \delta_j l)x\Big(-\frac{(3\lambda_m+4)}{(\lambda_m+1)}\cdot\frac{\Delta_j^m}{\delta_j} + G_{mj}^{14}\Big) \quad (2\delta_j\ \sinh\ \delta_j l)x$$

$$\Big(-\frac{(3\lambda_m+4)}{(\lambda_m+1)}\cdot\frac{\Delta_j^m}{\delta_j} + G_{mj}^{14}\Big) \quad (2\delta_j\ \cosh\ \delta_j l)x\Big(-\frac{(3\lambda_m+4)}{(\lambda_m+1)}\cdot\frac{G_{mj}^{19}}{\delta_j} + G_{mj}^{17}\Big)$$

$$(2\delta_j\ \sinh\ \delta_j l)x\Big(-\frac{(3\lambda_m+4)}{(\lambda_m+1)}\cdot\frac{G_{mj}^{19}}{\delta_j}\ G_{mj}^{17}\Big)\Big] \quad (EKNWN) \quad =$$

$$\Big[(2\alpha_j(-1)^j)x\Big(\frac{G_{mj}^5}{I_1(\alpha_j)}\Big) \quad (0)\Big] \quad (EALPH) \quad + \Big[(2\beta_j)x(G_{mj}^{10}) \quad (2\beta_j)x(G_{mj}^{10})\Big]$$

$$(EBETA) \qquad\qquad (A.18)$$

$$\Big[\Big(-\frac{8(\lambda_m+2)}{(\lambda_m+1)}\ lG_m^2\Big) \quad (0) \quad \Big(\frac{2\lambda_m}{(\lambda_m+1)}\ lG_m^2\Big) \quad (0) \quad (2\alpha_j(-1)^j)x$$

$$\Big(\frac{(\lambda_m+2)}{(\lambda_m+1)}\cdot\frac{G_{mj}^6}{\alpha_j I_o(\alpha_j)} + \frac{G_{mj}^7}{I_o(\alpha_j)}\Big) \quad (2\delta_j\ \sinh\ \delta_j l)x\Big(\frac{(\lambda_m+2)}{(\lambda_m+1)}\cdot\frac{G_{mj}^{15}}{\delta_j} + G_{mj}^{13}\Big)$$

$$(2\delta_j\ \cosh\ \delta_j l)x\Big(\frac{(\lambda_m+2)}{(\lambda_m+1)}\cdot\frac{G_{mj}^{15}}{\delta_j} + G_{mj}^{13}\Big) \quad (2\delta_j\ \sinh\ \delta_j l)x\Big(\frac{(\lambda_m+2)}{(\lambda_m+1)}\cdot\frac{G_{mj}^{18}}{\delta_j}$$

$$+ G_{mj}^{16}\Big) \quad (2\delta_j\ \cosh\ \delta_j l)x\Big(\frac{(\lambda_m+2)}{(\lambda_m+1)}\cdot\frac{G_{mj}^{18}}{\delta_j} + G_{mj}^{16}\Big)\Big] \quad (EKNWN) \quad =$$

$$\left[(0) \quad (2\gamma_j(-1)^j)x(-\frac{G^6_{mj}}{I_1(\alpha_j)})\right] \quad (EALPH) \quad + \left[(2\gamma_j \tanh\beta_j l)x(-G^{11}_{mj})\right.$$

$$\left.(2\beta_j)x(-\frac{G^{11}_{mj}}{\tanh\beta_j l})\right] \quad (EBETA) \qquad (A.20)$$

$$\left[(0) \quad (0) \quad (0) \quad (\frac{(3\lambda_m+4)}{(\lambda_m+1)} G^4_m) \quad (0) \quad (2\gamma_j)x(\frac{(\lambda_m+2)}{(\lambda_m+1)}\cdot\frac{G^6_{mj}}{\alpha_j I_o(\alpha_j)}\right.$$

$$+ \frac{G^7_{mj}}{I_o(\alpha_j)}) \quad (0) \quad (2\delta_j)x(\frac{(\lambda_m+2)}{(\lambda_m+1)}\cdot\frac{G^{15}_{mj}}{\delta_j} + G^{13}_{mj}) \quad (0)$$

$$\left.(2\delta_j)x(\frac{(\lambda_m+2)}{(\lambda_m+1)}\cdot\frac{G^{18}_{mj}}{\delta_j} + G^{16}_{mj})\right] \quad (EKNWN) \quad = \left[(0) \quad (2\alpha_j)x(-\frac{G^6_{mj}}{I_1(\alpha_j)})\right]$$

$$(EALPH) \quad + \left[(0) \quad (2\beta_j)x(-\frac{G^{11}_{mj}}{\sinh\beta_j l})\right] \quad (EBETA) \qquad (A.21)$$

(iii) <u>Equation on Interface (Bonded Region)</u>:

$$\left[(a^3X^1_m) \quad (aX^1_m) \quad (aX^3_m) \quad (a^3X^2_m) \quad (aX^4_m) \quad (X^6_{mj})x(\frac{aI_o(\alpha_j a)}{I_o(\alpha_j)}\right.$$

$$\left.(\Delta^m_j)x(\frac{aI_o(\alpha_j a)}{I_o(\alpha_j)}) \quad (X^7_{mj})x(aJ_o(\delta_j a) \quad (X^8_{mj})x(aJ_o(\delta_j a)\right] \quad (SKNWN) \quad =$$

$$\left[(a^3X^1_m) \quad (aX^1_m) \quad (aX^3_m) \quad (a^3X^2_m) \quad (X^6_{mj})x(\frac{aI_o(\alpha_j a)}{I_o(\alpha_j)}) \quad (\Delta^m_j)x(\frac{aI_o(\alpha_j a)}{I_o(\alpha_j)})\right.$$

$$(X^7_{mj})x(aJ_o(\delta_j a)) \quad (X^8_{mj})x(aJ_o(\delta_j a)) \quad (X^7_{mj})x(aY_o(\delta_j a))$$

$$\left.(X^8_{mj})x(aY_o(\delta_j a))\right] \quad (EKNWN) \quad + \left[(X^6_{mj})x(\frac{I_1(\alpha_j a)}{I_1(\alpha_j)}) \quad (\Delta^m_j)x(\frac{I_1(\alpha_j a)}{I_1(\alpha_j)})\right]$$

$$(EALPH) + \left[(X_{mj}^{9}) x (\frac{J_1(\beta_j a)}{\cosh \beta_j l}) \quad (X_{mj}^{10}) x (\frac{J_1(\beta_j a)}{\sinh \beta_j l}) \right] (EBETA)$$

$$- \left[(X_{mj}^{6}) x (\frac{I_1(\alpha_j a)}{I_1(\alpha_j)}) \quad (\Delta_j^m) x (\frac{I_1(\alpha_j a)}{I_1(\alpha_j)}) \right] (SALPH) \quad - \left[(X_{mj}^{9}) x (\frac{J_1(\beta_j a)}{\cosh \beta_j l}) \right.$$

$$\left. (X_{mj}^{10}) x (\frac{J_1(\beta_j a)}{\sinh \beta_j l}) \right] (SBETA) \tag{A.26}$$

$$\left[(-\frac{4(\lambda_f+2)}{(\lambda_f+1)} a^2 X_m^2 + \frac{8X_m^4}{3(\lambda_f+1)}) \quad (0) \quad (-\frac{1}{\lambda_f+1} a^2 X_m^2 - \frac{2}{3} \frac{\lambda_f}{(\lambda_f+1)} X_m^4) \right.$$

$$(-\frac{2(\lambda_f+2)}{(\lambda_f+1)} a^2 X_m^3 + \frac{2}{3} \frac{1}{(\lambda_f+1)} X_m^5 + \frac{(2\lambda_f+3)}{4(\lambda_f+1)} a^4 X_m^1) \quad (-\frac{3}{2(\lambda_f+1)} a^2 X_m^3$$

$$-\frac{1}{2(\lambda_f+1)} X_m^5 + \frac{3}{16} \frac{(\lambda_f+2)}{(\lambda_f+1)} a^4 X_m^1) \quad (\frac{aI_1(\alpha_j a)}{I_o(\alpha_j)} + \frac{2(\lambda_f+2)}{(\lambda_f+1)} \frac{I_o(\alpha_j a)}{I_o(\alpha_j)}$$

$$x(-\Delta_j^m) \quad (X_{mj}^{6}) x (\frac{aI_1(\alpha_j a)}{I_o(\alpha_j)} + \frac{2(\lambda_f+2)}{(\lambda_f+1)\alpha_j} \cdot \frac{I_o(\alpha_j a)}{I_o(\alpha_j)})$$

$$(X_{mj}^{8}) x (aJ_1(\delta_j a) - \frac{\lambda_m+2}{\lambda_m+1} \cdot \frac{2J_o(\delta_j a)}{\delta_j}) \quad (X_{mj}^{7}) x (aJ_1(\delta_j a)$$

$$\left. - \frac{\lambda_m+2}{\lambda_m+1} \cdot \frac{2J_o(\delta_j a)}{\delta_j}) \right] (SKNWN) \quad = \left[(-\frac{4(\lambda_m+2)}{(\lambda_m+1)} a^2 X_m^2 + \frac{8}{3} \frac{X_m^4}{(\lambda_m+1)}) \right.$$

$$(0) \quad (-\frac{1}{(\lambda_m+1)} a^2 X_m^2 - \frac{2\lambda_m}{3(\lambda_m+1)}) \quad (-\frac{2(\lambda_m+2)}{(\lambda_m+1)} a^2 X_m^3 + \frac{2}{(\lambda_m+1)} X_m^5$$

$$+ \frac{2\lambda_m+3}{4(\lambda_m+1)} a^4 X_m^1) \quad (-\Delta_j^m) x (\frac{aI_1(\alpha_j a)}{I_o(\alpha_j)} + \frac{2(\lambda_m+2)}{(\lambda_m+1)\alpha_j} \cdot \frac{I_o(\alpha_j a)}{I_o(\alpha_j)})$$

$$(X^6_{mj})x(\frac{aI_1(\alpha_j a)}{I_o(\alpha_j)} + \frac{2(\lambda_m+2)}{(\lambda_m+1)\alpha_j}\cdot\frac{I_o(\alpha_j a)}{I_o(\alpha_j)}) \quad (X^8_{mj})x(aJ_1(\xi_j a)$$

$$- \frac{\lambda_m+2}{\lambda_m+1}\cdot\frac{2J_o(\xi_j a)}{\xi_j}) \quad (X^7_{mj})x(aJ_1(\xi_j a) - \frac{\lambda_m+2}{\lambda_m+1}\cdot\frac{2J_o(\xi_j a)}{\xi_j})$$

$$(X^8_{mj})x(aY_1(\xi_j a) - \frac{\lambda_m+2}{\lambda_m+1}\cdot\frac{2Y_o(\xi_j a)}{\xi_j}) \quad (X^7_{mj})x(aY_1(\xi_j a)$$

$$- \frac{\lambda_m+2}{\lambda_m+1}\cdot\frac{2Y_o(\xi_j a)}{\xi_j})\Big] \text{ (EKNWN) } + \Big[(-\Delta^m_j)x(\frac{I_o(\alpha_j a)}{I_1(\alpha_j)}) \quad X^6_{mj}(\frac{I_o(\alpha_j a)}{I_1(\alpha_j)})\Big]$$

$$\text{(EALPH) } + \Big[(-X^{10}_{mj})x(\frac{J_o(\beta_j a)}{\cosh\beta_j l}) \quad (-X^9_{mj})x(\frac{J_o(\beta_j a)}{\sinh\beta_j l})\Big] \text{ (EBETA) } -$$

$$\Big[(-\Delta^m_j)x(\frac{I_o(\alpha_j a)}{I_1(\alpha_j)}) \quad (X^6_{mj})x(\frac{I_o(\alpha_j a)}{I_1(\alpha_j)})\Big] \text{ (SALPH) } - \Big[(-X^{10}_{mj})x(\frac{J_o(\beta_j a)}{\cosh\beta_j l})$$

$$(-X^9_{mj})x(\frac{J_o(\beta_j a)}{\sinh\beta_j l})\Big] \text{ (SBETA)} \qquad \text{(A.27)}$$

$$\Big[\zeta_F(\frac{2(\lambda_f+3)}{(\lambda_f+1)}a^2X^1_m + \frac{8\lambda_f}{(\lambda_f+1)}X^3_m) \quad \zeta_F(2(\lambda_f+1)X^1_m) \quad \zeta_F(-\frac{\lambda_f}{\lambda_f+1}a^2X^1_m$$

$$+ \frac{2(2\lambda_f+1)}{(\lambda_f+1)}X^3_m) \quad \zeta_F(\frac{2(\lambda_f+3)}{(\lambda_f+1)}a^2X^2_m + \frac{8\lambda_f}{3(\lambda_f+1)}X^4_m)$$

$$\zeta_F(-\frac{3\lambda_f}{(\lambda_f+1)}a^2X^2_m + \frac{2(2\lambda_f+1)}{3(\lambda_f+1)}X^4_m) \quad (2\zeta_F\alpha_j X^6_{mj})(\frac{aI_1(\alpha_j a)}{I_o(\alpha_j)}$$

$$+ \frac{1}{\alpha_j(\lambda_f+1)}\cdot\frac{I_o(\alpha_j a)}{I_o(\alpha_j)}) \quad (2\zeta_F\alpha_j\Delta^m_j)(\frac{aI_1(\alpha_j a)}{I_o(\alpha_j)} + \frac{I_o(\alpha_j a)}{\alpha_j(\lambda_f+1)I_o(\alpha_j)})$$

$(-2\zeta_F \xi_j X^7_{mj})(aJ_1(\xi_j a) - \frac{1}{(\lambda_f+1)} \cdot \frac{J_o(\xi_j a)}{\xi_j})$ $(-2\zeta_F \delta_j X^8_{mj})$

$(aJ_1(\delta_j a) - \frac{1}{(\lambda_f+1)} \cdot \frac{J_o(\xi_j a)}{\xi_j})\Big]$ (SKNWN) $= \Big[(\frac{2(\lambda_m+3)}{(\lambda_m+1)} a^2 X^1_m$

$+ \frac{8\lambda_m}{(\lambda_m+1)} X^3_m)$ $(2(\lambda_m+1)X^1_m)$ $(-\frac{\lambda_m}{\lambda_m+1} a^2 X^1_m + \frac{2(2\lambda_m+1)}{(\lambda_m+1)} X^3_m)$

$(\frac{2(\lambda_m+3)}{(\lambda_m+1)} a^2 X^2_m + \frac{8\lambda_m}{3(\lambda_m+1)} X^4_m)$ $(2\alpha_j X^6_{mj})(\frac{aI_1(\alpha_j a)}{I_o(\alpha_j)} +$

$\frac{1}{(\lambda_m+1)\alpha_j} \cdot \frac{I_o(\alpha_j a)}{I_o(\alpha_j)})$ $(2\alpha_j \Delta^m_j)(\frac{aI_1(\alpha_j a)}{I_o(\alpha_j)} + \frac{1}{(\lambda_m+1)\ j} \cdot \frac{I_o(\alpha_j a)}{I_o(\alpha_j)})$

$(-2\xi_j X^7_{mj})(aJ_1(\xi_j a) - \frac{1}{(\lambda_m+1)} \cdot \frac{J_o(\lambda_j a)}{\delta_j})$ $(-2\delta_j X^8_{mj})(aJ_1(\delta_j a)$

$- \frac{1}{(\lambda_m+1)} \cdot \frac{J_o(\delta_j a)}{\delta_j})$ $(-2\delta_j X^7_{mj})(aY_1(\xi_j a) - \frac{1}{(\lambda_m+1)} \cdot \frac{Y_o(\xi_j a)}{\delta_j})$

$(-2\delta_j X^8_{mj})(aY_1(\xi_j a) - \frac{1}{(\lambda_m+1)} \cdot \frac{Y_o(\delta_j a)}{\delta_j})\Big]$ (EKNWN) $+$

$\Big[(2\alpha_j X^6_{mj})(\frac{I_o(\alpha_j a)}{I_1(\alpha_j)} - \frac{1}{\alpha_j a} \cdot \frac{I_1(\alpha_j a)}{I_1(\alpha_j)})$ $(2\alpha_j \ ^m_j)(\frac{I_o(\alpha_j a)}{I_1(\alpha_j)} -$

$\frac{1}{\alpha_j a} \cdot \frac{I_1(\alpha_j a)}{I_1(\alpha_j)})\Big]$ (EALPH) $+ \Big[(X^9_{mj}\ \beta_j/\cosh \beta_j 1)(J_o(\beta_j a) - \frac{J_1(\beta_j a)}{a\beta_j})$

$(2X^{10}_{mj}\ \beta_j/\sinh \beta_j 1)(J_o(\beta_j a) - \frac{J_1(\beta_j a)}{a\beta_j})\Big]$ (EBETA) $-$

$$\Big[(2\zeta_F\alpha_j X^6_{mj})\left(\frac{I_o(\alpha_j a)}{I_1(\alpha_j)} - \frac{I_1(\alpha_j a)}{\alpha_j a I_1(\alpha_j)}\right) \quad (2\zeta_F\alpha_j \Delta^m_j)\left(\frac{I_o(\alpha_j a)}{I_1(\alpha_j)} - \frac{I_1(\alpha_j a)}{j a I_1(\alpha_j)}\right)\Big] \quad \text{(SALPH)} \quad - \Big[(2\beta_j/\cosh\beta_j 1)\left(J_o(\beta_j a) - \frac{J_1(\beta_j a)}{a\beta_j}\right)\zeta_F X^9_{mj}$$

$$(2\beta_j/\sinh\beta_j 1)\left(J_o(\beta_j a) - \frac{J_1(\beta_j a)}{a\beta_j}\right)\zeta_F X^{10}_{mj}\Big] \quad \text{(SBETA)} \qquad \text{(A.29)}$$

$$\Big[\zeta_F\left(-\frac{8(\lambda_f+2)}{(\lambda_f+1)}aX^2_m\right) \quad (0) \quad \zeta_F\left(\frac{2\lambda_f}{(\lambda_f+1)}aX^2_m\right) \quad \zeta_F\left(\frac{3\lambda_f+4}{\lambda_f+1}a^3X^1_m - \frac{4(\lambda_f+2)}{(\lambda_f+1)}aX^3_m\right) \quad \zeta_F\left(\frac{3(\lambda_f+2)}{4(\lambda_f+1)}a^3X^1_m + \frac{3\lambda_f}{(\lambda_f+1)}aX^3_m\right) \quad (-2\zeta_F\alpha_j\Delta^m_j)\,x$$

$$\left(\frac{aI_o(\alpha_j a)}{I_o(\alpha_j)} + \frac{(\lambda_f+2)}{(\lambda_f+1)\alpha_j}\cdot\frac{I_1(\alpha_j a)}{I_o(\alpha_j)}\right) \quad (2\zeta_F\alpha_j X^6_{mj})\left(\frac{aI_o(\alpha_j a)}{I_o(\alpha_j)} + \frac{(\lambda_f+2)}{(\lambda_f+1)\alpha_j}\cdot\frac{I_1(\alpha_j a)}{I_o(\alpha_j)}\right) \quad (2\zeta_F\delta_j X^8_{mj})\left(aJ_o(\delta_j a) + \frac{\lambda_f+2}{\lambda_f+1}\cdot\frac{J_1(\delta_j a)}{\delta_j}\right)$$

$$(2\zeta_F\delta_j X^7_{mj})\left(aJ_o(\delta_j a) + \frac{\lambda_f+2}{\lambda_f+1}\cdot\frac{J_1(\delta_j a)}{\delta_j}\right)\Big] \quad \text{(SKNWN)} \quad =$$

$$\Big[\left(-\frac{8(\lambda_m+2)}{(\lambda_m+1)}aX^2_m\right) \quad (0) \quad \left(\frac{2\lambda_m}{(\lambda_m+1)}aX^2_m\right) \quad \left(\frac{3\lambda_m+4}{\lambda_m+1}a^3X^1_m - \frac{4(\lambda_m+2)}{(\lambda_m+1)}aX^3_m\right) \quad (-2\alpha_j\Delta^m_j)\left(\frac{aI_o(\alpha_j a)}{I_o(\alpha_j)} + \frac{(\lambda_m+2)}{(\lambda_m+1)\alpha_j}\cdot\frac{I_1(\alpha_j a)}{I_o(\alpha_j)}\right)$$

$$(2\alpha_j X^6_{mj})\left(\frac{aI_o(\alpha_j a)}{I_o(\alpha_j)} + \frac{(\lambda_m+2)}{(\lambda_m+1)\alpha_j}\cdot\frac{I_1(\alpha_j a)}{I_o(\alpha_j)}\right) \quad (2\delta_j X^8_{mj})(aJ_o(\delta_j a)$$

$$+ \frac{\lambda_m+2}{\lambda_m+1} \cdot \frac{J_1(\delta_j a)}{\delta_j}) \quad (2\delta_j X^7_{mj})(aJ_o(\delta_j a) + \frac{\lambda_m+2}{\lambda_m+1} \cdot \frac{J_1(\delta_j a)}{\delta_j})$$

$$(2\delta_j X^8_{mj})(aY_o(\delta_j a) + \frac{\lambda_m+2}{\lambda_m+1} \cdot \frac{Y_1(\delta_j a)}{\delta_j}) \quad (2\delta_j X^7_{mj})(aY_o(\delta_j a) +$$

$$\frac{\lambda_m+2}{\lambda_m+1} \cdot \frac{Y_1(\delta_j a)}{\delta_j})\Big] \quad (\text{EKNWN}) \quad + \Big[(-2\alpha_j A^m_j)(\frac{I_1(\alpha_j a)}{I_1(\alpha_j)})$$

$$(2\alpha_j X^6_{mj})(\frac{I_1(\alpha_j a)}{I_1(\alpha_j)})\Big] \quad (\text{EALPH}) \quad + \Big[(2X^{10}_{mj}\, \beta_j/\cosh \beta_j 1)(J_1(\beta_j a))$$

$$(2X^9_{mj}\, \beta_j/\sinh \beta_j 1)(J_1(\beta_j a))\Big] \quad (\text{EBETA}) \quad - \Big[(-2\zeta_F \alpha_j A^m_j)(\frac{I_1(\alpha_j a)}{I_1(\alpha_j)})$$

$$(2\zeta_F \alpha_j X^6_{mj})(\frac{I_1(\alpha_j a)}{I_1(\alpha_j)})\Big] \quad (\text{SALPH}) \quad - \Big[(2\zeta_F X^{10}_{mj}\, \beta_j/\cosh \beta_j 1)(J_1(\beta_j a))$$

$$(2\zeta_F X^9_{mj}\, \beta_j/\sinh \beta_j 1)(J_1(\beta_j a))\Big] \quad (\text{SBETA}) \qquad (A.28)$$

(iv) <u>Equations on Interface (Debonded Region):</u>

$$\Big[(\frac{2(\lambda_f+3)}{(\lambda_f+1)} a^2 + \frac{8\lambda_f}{(\lambda_f+1)} z^2_m) \quad (2(\lambda_f+1) \quad (-\frac{\lambda_f}{\lambda_f+1} a^2 +$$

$$\frac{2(2\lambda_f+1)}{(\lambda_f+1)} z^2_m) \quad (\frac{2(\lambda_f+3)}{(\lambda_f+1)} a^2 z_m + \frac{8\lambda_f}{3(\lambda_f+1)} z^3_m) \quad (-\frac{3\lambda_f}{\lambda_f+1} a^2 z_m$$

$$+ \frac{2(2\lambda_f+1)}{3(\lambda_f+1)} z^3_m) \quad (2\alpha_j \cos \alpha_j z_m)(\frac{aI_1(\alpha_j a)}{I_o(\alpha_j)} + \frac{I_o(\alpha_j a)}{\alpha_j(\lambda_f+1) I_o(\alpha_j)})$$

$$(2\alpha_j \sin \alpha_j z_m)(\frac{aI_1(\alpha_j a)}{I_o(\alpha_j)} + \frac{I_o(\alpha_j a)}{\alpha_j I_o(\alpha_j)(\lambda_f+1)})$$

$$(-2\delta_j \cosh \delta_j z_m)(aJ_1(\delta_j a) - \frac{1}{(\lambda_f+1)}\cdot\frac{J_o(\delta_j a)}{\delta_j})$$

$$(-2\delta_j \sinh \delta_j z_m)(aJ_1(\delta_j a) - \frac{1}{(\lambda_f+1)}\cdot\frac{J_o(\delta_j a)}{\delta_j})\Big] \quad (SKNWN)$$

$$+\Big[(2\alpha_j \cos \alpha_j z_n)(\frac{I_o(\alpha_j a)}{I_o(\alpha_j)} - \frac{I_1(\alpha_j a)}{\alpha_j a\, I_1(\alpha_j)}) \quad (2\alpha_j \sin \alpha_j z_m)x$$

$$(\frac{I_o(\alpha_j a)}{I_1(\alpha_j)} - \frac{I_1(\alpha_j a)}{\alpha_j a\, I_1(\alpha_j)})\Big] \quad (SALPH) \quad +\Big[(2\beta_j \frac{\cosh \beta_j z_m}{\cosh \beta_j l})(J_o(\beta_j a)$$

$$-\frac{J_1(\beta_j a)}{\beta_j a}) \quad (2\beta_j \frac{\sinh \beta_j z_m}{\sinh \beta_j l})(J_o(\beta_j a) - \frac{J_1(\beta_j a)}{\beta_j a})\Big] \quad (SBETA) = (\Omega) \tag{A.36}$$

$$\Big[-\frac{8(\lambda_f+2)}{(\lambda_f+1)} az_m) \quad (0) \quad (\frac{2\lambda_f}{\lambda_f+1} az_m) \quad (\frac{3\lambda_f+4}{\lambda_f+1} a^3 - \frac{4(\lambda_f+1)}{(\lambda_f+1)} az_m^2)$$

$$(\frac{3(\lambda_f+2)}{4(\lambda_f+1)} a^3 + \frac{3\lambda_f}{(\lambda_f+1)} az_m^2) \quad (-2\alpha_j \sin \alpha_j z_m)(\frac{aI_o(\alpha_j a)}{I_o(\alpha_j)} +$$

$$\frac{(\lambda_f+2)}{(\lambda_f+1)\alpha_j}\cdot\frac{I_1(\alpha_j a)}{I_o(\alpha_j)}) \quad (2\alpha_j \cos \alpha_j z_m)(\frac{aI_o(\alpha_j a)}{I_o(\alpha_j)} +$$

$$\frac{(\lambda_f+2)}{(\lambda_f+1)\alpha_j}\cdot\frac{I_1(\alpha_j a)}{I_o(\alpha_j)}) \quad (2\delta_j \sinh \delta_j z_m)(aJ_o(\delta_j a) + \frac{\lambda_f+2}{\lambda_f+1}\cdot\frac{J_1(\delta_j a)}{\delta_j})$$

$$(2\delta_j \cosh \delta_j z_m)(aJ_o(\delta_j a) + \frac{\lambda_f+2}{\lambda_f+1}\cdot\frac{J_1(\delta_j a)}{\delta_j})\Big] \quad (SKNWN) \quad +$$

$$\Big[(-2\alpha_j \sin \alpha_j z_m)(\frac{I_1(\alpha_j a)}{I_1(\alpha_j)}) \quad (2\alpha_j \cos \alpha_j z_m)(\frac{I_1(\alpha_j a)}{I_1(\alpha_j)})\Big] \quad (SALPH)$$

$$+\left[(2\beta_j \frac{\sinh \beta_j z_m}{\cosh \beta_j l})(J_1(\beta_j a) \quad (2\beta_j \frac{\cosh \beta_j z_m}{\sinh \beta_j l})(J_1(\beta_j a))\right]$$

$$(\text{SBETA}) = (0) \qquad (A.38)$$

$$\left[(\frac{-2(\lambda_m+3)}{(\lambda_m+1)} a^2 + \frac{8\lambda_m}{\lambda_m+1} z_m^2) \quad (2(\lambda_m+1) \quad (\frac{-\lambda_m}{\lambda_m+1} a^2 + \frac{2(2\lambda_m+1)}{\lambda_m+1} z_m^2)\right.$$

$$(\frac{2(\lambda_m+3)}{(\lambda_m+1)} a^2 z_n + \frac{8\lambda_m}{3(\lambda_m+1)} z_m^3) \quad (2\alpha_j \cos \alpha_j z_m)(\frac{aI_1(\alpha_j a)}{I_o(\alpha_j)} +$$

$$\frac{I_o(\alpha_j a)}{\alpha_j(\lambda_m+1)I_o(\alpha_j)}) \quad (2\alpha_j \sin \alpha_j z_m)(\frac{aI_1(\alpha_j a)}{I_o(\alpha_j)} + \frac{I_o(\alpha_j a)}{\alpha_j(\lambda_m+1)I_o(\alpha_j)})$$

$$(-2\delta_j \cosh \delta_j z_m)(aJ_1(\delta_j a) - \frac{J_o(\delta_j a)}{(\lambda_m+1)\delta_j}) \quad (-2\delta_j \sinh \delta_j z_m) x$$

$$(aJ_1(\delta_j a) - \frac{J_o(\delta_j a)}{(\lambda_m+1)\delta_j}) \quad (-2\delta_j \cosh \delta_j z_m)(aY_1(\delta_j a) -$$

$$\left.\frac{Y_o(\delta_j a)}{(\lambda_m+1)\delta_j}) \quad (-2\delta_j \sinh \delta_j z_m)(aY_1(\delta_j a) - \frac{Y_o(\delta_j a)}{(\lambda_m+1)\delta_j})\right]$$

$$(\text{EKNWN}) + \left[(2\alpha_j \cos \alpha_j z_m)(\frac{I_o(\alpha_j a)}{I_1(\alpha_j)} - \frac{I_1(\alpha_j a)}{\alpha_j a I_1(\alpha_j)})\right.$$

$$\left.(2\alpha_j \sin \alpha_j z_m)(\frac{I_o(\alpha_j a)}{I_1(\alpha_j)} - \frac{I_1(\alpha_j a)}{\alpha_j a I_1(\alpha_j)})\right] \quad (\text{EALPH}) +$$

$$\left[(2\beta_j(\cosh \beta_j z_m/\cosh \beta_j l))(J_o(\beta_j a) - \frac{J_1(\beta_j a)}{\beta_j a})\right.$$

$$\left.(2\beta_j \frac{\sinh \beta_j z_m}{\sinh \beta_j l})(J_o(\beta_j a) - \frac{J_1(\beta_j a)}{\beta_j a})\right] \quad (\text{EBETA}) = (0)$$

$$(A.35)$$

$$\left(-\frac{8(\lambda_m+2)}{(\lambda_m+1)}\,az_m\right) \quad (0) \quad \left(\frac{2\lambda_m}{\lambda_m+1}\,az_m\right) \quad \left(\frac{3\lambda_m+4}{\lambda_m+1}a^3 - \frac{4(\lambda_m+2)}{(\lambda_m+1)}\,az_m^2\right)$$

$$(-2\alpha_j \sin\alpha_j z_m)\left(\frac{aI_o(\alpha_j a)}{I_o(\alpha_j)} + \frac{\lambda_m+2}{\lambda_m+1}\cdot\frac{I_1(\alpha_j a)}{\alpha_j I_o(\alpha_j)}\right)$$

$$(2\alpha_j \cos\alpha_j z_m)\left(\frac{aI_o(\alpha_j a)}{I_o(\alpha_j)} + \frac{\lambda_m+2}{\lambda_m+1}\cdot\frac{I_1(\alpha_j a)}{\alpha_j I_o(\alpha_j)}\right)$$

$$(2\delta_j \sinh\delta_j z_m)\left(aJ_o(\delta_j a) + \frac{\lambda_m+2}{\lambda_m+1}\cdot\frac{J_1(\delta_j a)}{\delta_j}\right)$$

$$(2\delta_j \cosh\delta_j z_m)\left(aJ_o(\delta_j a) + \frac{\lambda_m+2}{\lambda_m+1}\cdot\frac{J_1(\delta_j a)}{\delta_j}\right)$$

$$(2\delta_j \sinh\delta_j z_m)\left(aY_o(\delta_j a) + \frac{\lambda_m+2}{\lambda_m+1}\cdot\frac{Y_1(\delta_j a)}{\delta_j}\right)$$

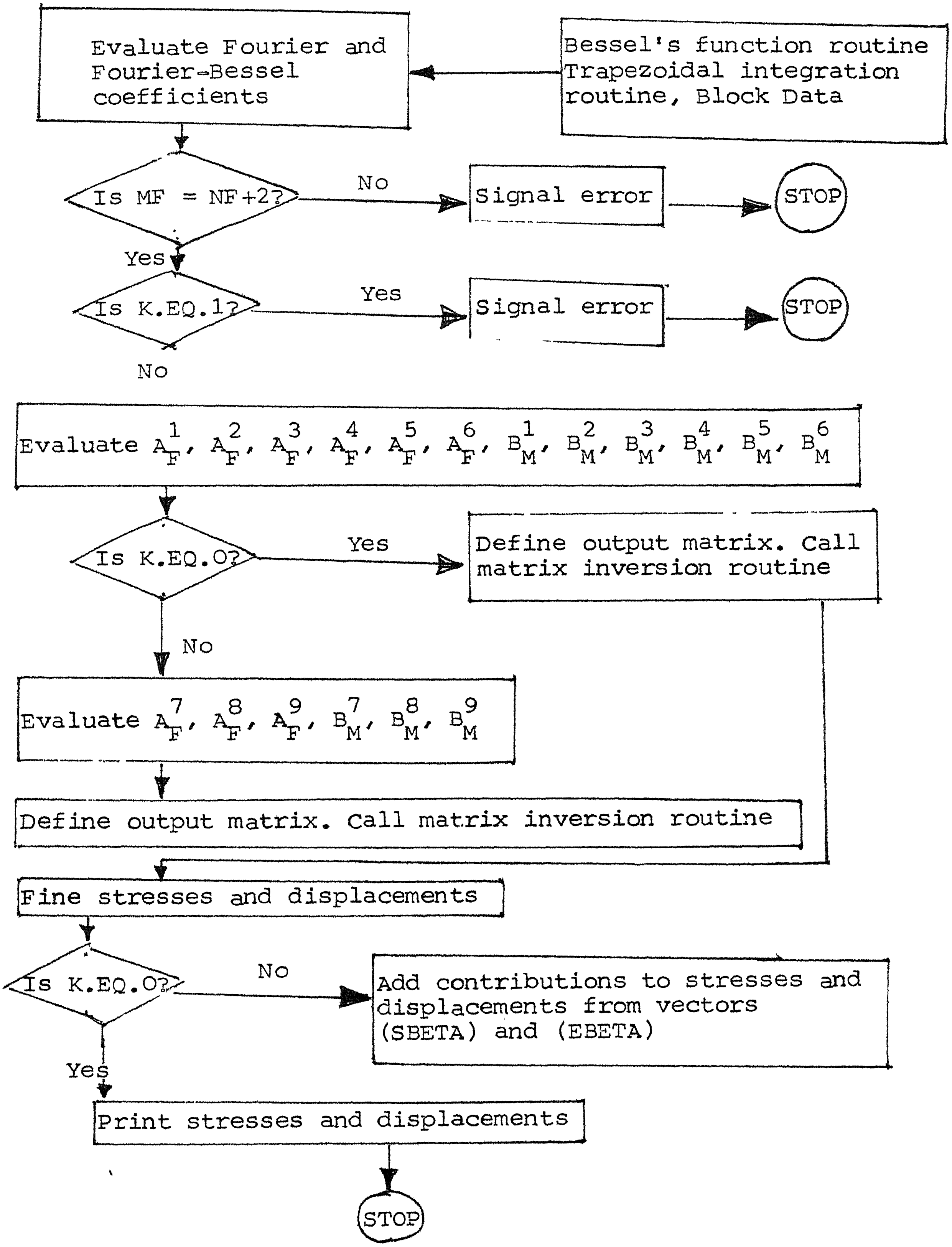
$$(2\delta_j \cosh\delta_j z_m)\left(aY_o(\delta_j a) + \frac{\lambda_m+2}{\lambda_m+1}\cdot\frac{Y_1(\delta_j a)}{\delta_j}\right)\Big] \quad (\text{EKNWN}) \quad +$$

$$\left[(-2\alpha_j \sin\alpha_j z_m)\left(\frac{I_1(\alpha_j a)}{I_1(\alpha_j)}\right) \quad (2\alpha_j \cos\alpha_j z_m)\left(\frac{I_1(\alpha_j a)}{I_1(\alpha_j)}\right)\right]$$

$$(\text{EALPH}) \quad + \Big[(2\beta_j\,\frac{\sinh\beta_j z_m}{\cosh\beta_j 1})(J_1(\beta_j a))$$

$$(2\beta_j\,\frac{\cosh\beta_j z_m}{\sinh\beta_j 1})(J_1(\beta_j a))\Big] \quad (\text{EBETA}) \quad = \quad (0) \qquad (\text{A.37})$$

APPENDIX III: FLOW CHART



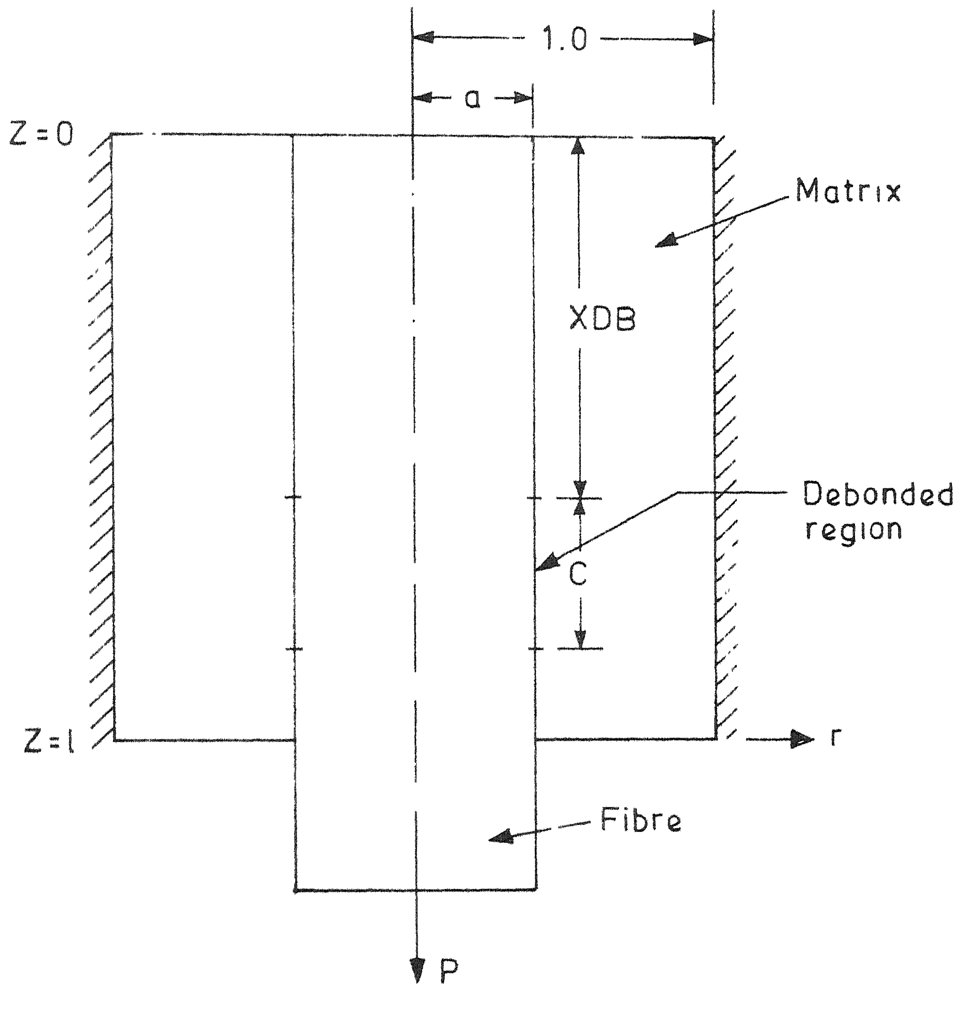


FIG. 2.1 ELASTIC MODEL

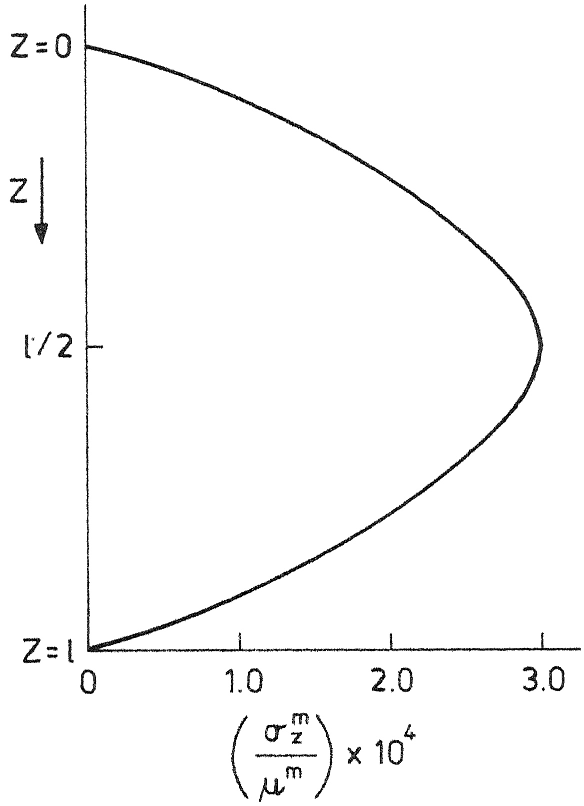


Fig. 3.1 Variation of nondimensional axial stress with nondimensional length.

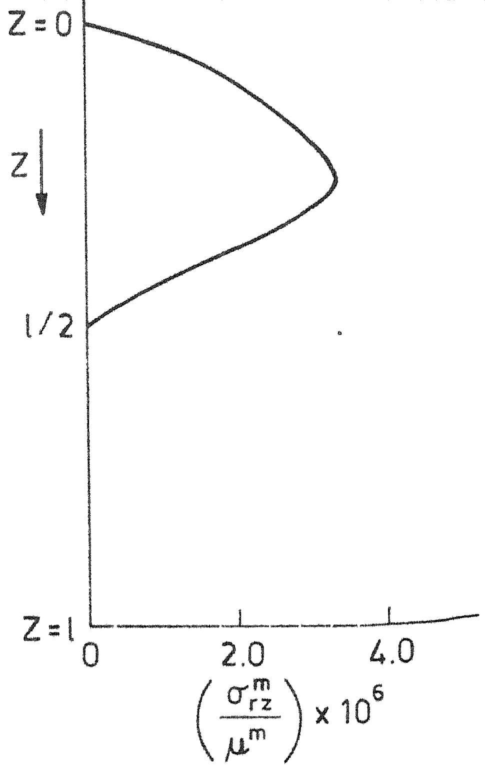


Fig. 3.2 Variation of nondimensional shear stress with nondimensional length.

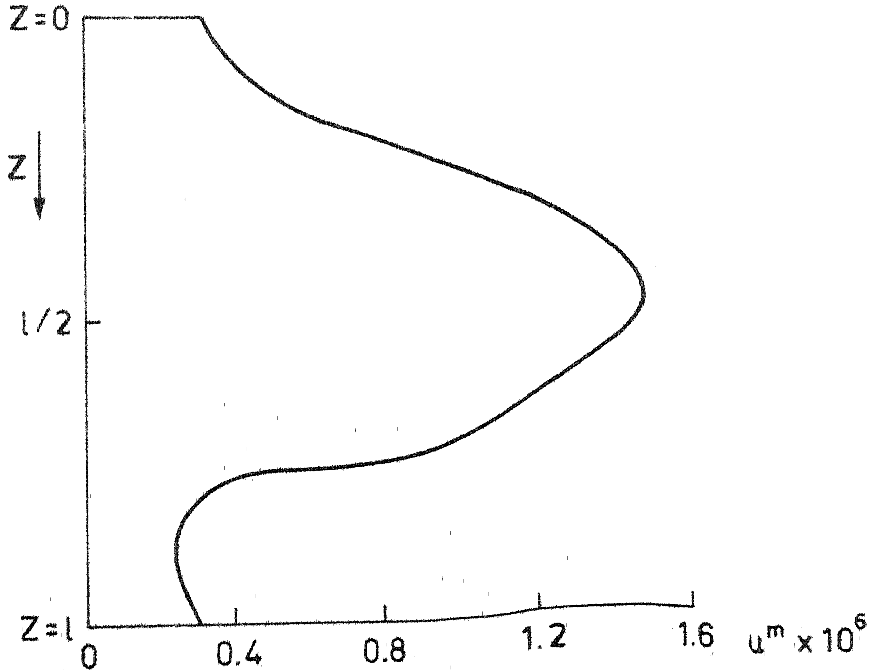


Fig. 3.3 Variation of nondimensional radial displacement with nondimensional length.